Always Remember:

There are four great realms of mechanics:

Newtonian Mechanics, Special Relativity, Quantum Mechanics, Quantum Field Theory

For everyday life, Newtonian mechanics was found to be inadequate.

But for objects moving with high speeds, near the speed of light, it is incorrect and must be replaced with Special Relativity introduced by Einstein in 1905. For objects that are extremely small near the size of atoms, it fails and explained by Quantum Mechanics. For objects that are both very fast and very small as it is common in modern particle physics, a mechanics that combines relativity and quantum principles in order is called relativistic quantum mechanics or quantum field theory.

The laws of classical electrodynamics were discovered by Franklin, Coulomb, Ampere, Faraday and others. But the person who developed and added all work in compact and consistent form is James Clerk Maxwell.

In the beginning, electricity and magnetism were two separate subjects. One dealt with glass rods and cat's fur, pith balls, batteries, current, electrolysis and lightning; the other with bar magnets, iron filling, compass needles, north pole etc.

In 1820, Oersted noticed that an electric current could deflect magnetic compass needle. Then Ampere postulated that all magnetic phenomenon are due to electric charges in motion. In 1831, Faraday discovered that a moving magnet generates an electric current. At the same time Maxwell and Lorentz developed correlations between them and two subjects: electricity and magnetism became single subject electromagnetism. Faraday assumed that the light is electrical in nature. Maxwell's theory proved it and in optics the study of lenses, mirrors, prisms, interference, and diffraction was incorporated in to electromagnetism. Hertz presented the experimental confirmation of Maxwell's theory in 1888.

The fundamental theorem for gradients:

$$\int_a^b (\nabla T) \cdot d\boldsymbol{l} = T(\boldsymbol{b}) - T(\boldsymbol{a})$$

The fundamental theorem for divergences:

$$\int (\nabla \cdot \boldsymbol{v}) d\tau = \oint \boldsymbol{v} \cdot d\boldsymbol{a}$$

It is called the divergence theorem or Gauss's theorem or Green's theorem. It says that the integral of derivative over a region (i.e. divergence) is equal to the value of the function (i.e. volume) at the boundary (i.e. surface that covers or bounds the volume)

Boundary of a line is just two end points and boundary of a volume is a closed surface.

The fundamental theorem for curls:

$$\int (\nabla \times \boldsymbol{v}) da = \oint \boldsymbol{v} \cdot d\boldsymbol{l}$$

It is called Stokes' theorem. The integral of a derivative (i.e. curl) over a region (i.e. patch of surface) is equal to the value of the function at the boundary (i.e. the perimeter of the patch).

1. The force on a test charge Q due to a single point charge q which is at rest at a distance r is given by **Coulomb's law**. The force is proportional to the product of the charges and inversely proportional to the square of the separation distance.

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\,\hat{\boldsymbol{r}} \quad (1)$$

The constant ϵ_0 is called the permittivity of free space. In SI units, where force is in Newton (N), distance in meters (m), charge in coulombs (C),

$$\epsilon_0 = 8.85 \times 10^{-12}\frac{C^2}{N \cdot m^2}$$

The electric field due to point charge or single charge is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\boldsymbol{r}} \quad (2)$$

Such that

$$\boldsymbol{F} = Q\boldsymbol{E} \quad (3)$$

2. If we have several point charges $q_1, q_2, \ldots, q_n$ at distances $r_1, r_2, \ldots, r_n$ from Q, the total force on Q is given by

$$\boldsymbol{F} = \boldsymbol{F_1} + \boldsymbol{F_1} + \cdots = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{r_1^2}\,\hat{\boldsymbol{r}}_1 + \frac{q_2 Q}{r_2^2}\,\hat{\boldsymbol{r}}_2 + \cdots\right) = \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^2}\,\hat{\boldsymbol{r}}_1 + \frac{q_2}{r_2^2}\,\hat{\boldsymbol{r}}_2 + \cdots\right) \quad (4)$$

$$\boldsymbol{F} = Q\boldsymbol{E}$$

Where:

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\,\hat{\boldsymbol{r}}_i \quad (5)$$

$\boldsymbol{E}$ is called the electric field of the source charges. Here we assumed that the source of the field is a collection of discrete point charges q_i or the charge distribution is discrete.

3. If the charge distribution is distributed continuously over some region, the summation is replaced with an integral and written as

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{1}{r^2}\,\hat{\boldsymbol{r}}\,dq \quad (6)$$

4. If the charge is spread out along a line, with "charge per unit length" is λ then $dq = \lambda\,dl'$, where dl' is an element of length along the line then the electric field of a line charge is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\boldsymbol{r}')}{r^2}\,\hat{\boldsymbol{r}}\,dl' \quad (7)$$

5. If the charge is spread out over a surface, with "charge per unit area" is σ then $dq = \sigma\,da'$, where da' is an element of area on the surface then the electric field for a surface charge is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\boldsymbol{r}')}{r^2}\,\hat{\boldsymbol{r}}\,da' \quad (8)$$

6. If the charge feels a volume, with "charge per unit volume" is ρ then $dq = \rho\,d\tau'$, where $d\tau'$ is an element of volume then the electric field of a volume charge is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{r}')}{r^2}\,\hat{\boldsymbol{r}}\,d\tau' \quad (9)$$

7. Flux of $\boldsymbol{E}$ through a surface S is given by

$$\Phi_E = \int \boldsymbol{E}\cdot\boldsymbol{da} \quad (10)$$

It is a measure of the number of field lines passing through it. Flux is proportional to the number of field lines or to the density of field lines.

8. For any close surface

$$\oint \boldsymbol{E} \cdot \boldsymbol{da} = \frac{1}{\epsilon_0} Q_{enc} \quad (11)$$

This is quantitative statement of Gauss's law. As it stands, Gauss's law in an integral equation, it can be turn into a differential one by applying the divergence theorem.

$$\oint \boldsymbol{E} \cdot \boldsymbol{da} = \int (\nabla \cdot \boldsymbol{E})\, d\tau \quad (12)$$

Q_{enc} in terms of charge density

$$Q_{enc} = \int \rho\, d\tau$$

So

$$\int (\nabla \cdot \boldsymbol{E})\, d\tau = \int \frac{\rho}{\epsilon_0}\, d\tau$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} = \frac{1}{\epsilon_0}\rho$$

This is differential form of Gauss's law.

4. we know that

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\boldsymbol{r}}$$

the line integral of this field from some point a to some another point b is given by

$$\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l}$$

In spherical coordinates

$$d\boldsymbol{l} = dr\,\hat{\boldsymbol{r}} + rd\theta\,\widehat{\boldsymbol{\theta}} + rsin\theta d\emptyset\,\widehat{\emptyset}$$

$$\boldsymbol{E} \cdot d\boldsymbol{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\, dr$$

Therefore

$$\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l} = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}\,dr = -\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r}\right)_{r_a}^{r_b} = -\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_b}-\frac{q}{r_a}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a}-\frac{q}{r_b}\right)$$

Here r_a is the distance from the origin to the point $\boldsymbol{a}$ and r_b is the distance to $\boldsymbol{b}$.

The integral around a closed path is zero as $r_a = r_b$

$$\oint \boldsymbol{E}\cdot d\boldsymbol{l} = 0$$

By applying Stoke's theorem

$$\nabla\times\boldsymbol{E} = 0 \quad (13)$$

Because $\nabla\times\boldsymbol{E} = 0$, the line integral of $\boldsymbol{E}$ around any closed loop is zero, and line integral is independent of path, we can define a function

$$V(r) = -\int_0^r \boldsymbol{E}\cdot d\boldsymbol{l}$$

Here O is some standard reference point V then depends only on the point $\boldsymbol{r}$. It is called the electric potential. The electric field can be written as the gradient of a scalar potential.

$$\boldsymbol{E} = -\nabla V$$

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0}, \nabla\times\boldsymbol{E} = 0$$

$$\nabla\cdot\boldsymbol{E} = \nabla\cdot(-\nabla V) = -\nabla^2 V$$

i.e. divergence of $\boldsymbol{E}$ is the Laplacian of V. Gauss's law says that

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

This equation is known as **Poisson's equation**. In regions where there is no charge, so that $\rho = 0$, Poisson's equation reduces to **Laplace's equation**.

$$\nabla^2 V = 0$$

Setting the reference point at infinity, the potential of a point charge q at the origin

$$V(r) = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr' = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r'}\right)_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

5. Now the potential due to a point charge is

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

Similarly, the potential of a collection of charges

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$$

Similarly, for continuous distribution

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{1}{r}\,dq$$

Similarly, for line charge

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\boldsymbol{r}')}{r}dl' \quad (7)$$

Similarly, for surface charge

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\boldsymbol{r}')}{r}\,da' \quad (8)$$

Similarly, for volume charge

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{r}')}{r}\,d\tau' \quad (9)$$

1. Conductors:

1.1 Basic Properties:

We know that in insulators like glass or rubber, each electron is attached to a particular atom. Where as in metallic conductor, one or more electrons per atom are free to roam about at will through the material. In liquid conductors like salt water it is the ions that move. A perfect conductor would be a material which contain an unlimited supply of completely free charges. In real life there are no perfect conductors but there are many substances they come close to perfect conductor. The basic electrostatic properties of ideal conductors are:

(i) $\boldsymbol{E} = 0$ inside a conductor:

If there were any field, those free charges would move, and it wouldn't be electrostatics any more.

When we put a conductor into an external electric field $\boldsymbol{E}_0$ as shown in Figure: 1

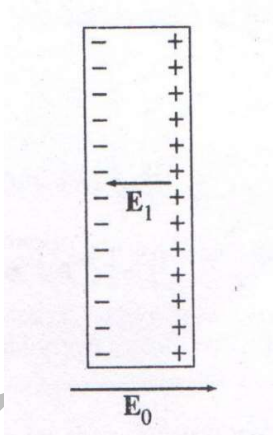

Figure: 1

Initially this will drive any free positive charges to the right, and negative charges to the left. (In practice it's only negative charges i.e. electrons that do the moving, but when electrons move or depart the right side is left with a net positive charge - the stationary nuclei - so it doesn't really matter which charges move; the effect is the same.)

When they come to the edge of the material, the charges pile up: plus charges on the right side, minus charges on the left side. Now these **induced charges produce a field of their own, E_1**, which, as you can see from the Figure: 1, is in the **opposite direction to E_0**. It means that the field of the induced charges **tends to cancel off the original field.** Charge will continue to flow until this cancellation is complete, and the resultant field inside the conductor is precisely zero. (outside the conductor the field is not zero, for here E_0 and E_1 do not cancel). In fact the whole process is practically instantaneous.

(ii) $\rho = 0$ inside the conductor.

We have Gauss's law as

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \qquad if\ \boldsymbol{E} = 0, \qquad \rho = 0$$

There is still charge around, but exactly as much plus charge as minus, so the net charge density in the interior is zero.

(iii) Any net charge resides on the surface.

(iv) A conductor is an equipotential.

Consider any two points a and b within or at the surface of a given conductor

$$V(a) - V(b) = -\int_a^b E \cdot dl = 0$$

and thus $V(a) = V(b)$.

(v) E is perpendicular to the surface, just outside the conductor: Otherwise as in (i) charge will immediately flow around the surface until it kills off the tangential component as shown in Figure: 2 perpendicular to the surface, the charge cannot flow, of course, since it is confined to the conducting object.

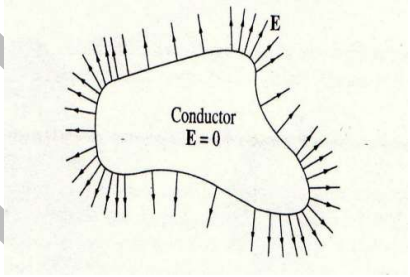

Figure: 2

It is strange that the charge on a conductor flows to the surface. Because of their mutual repulsion, the charges naturally spread out as much as possible, but for all of them to go to the surface seems like a waste of the interior space.

The problem can also be phrased in terms of energy. Like any other free dynamical system, the charge on a conductor will seek the configuration that minimizes its potential energy.
What property asserts is that the electrostatic energy of a solid object (with specified shape and total charge) is a minimum when that charge is spread over the surface. For instant, the energy

of a sphere is $\left(\frac{1}{8\pi\epsilon_0}\frac{q^2}{R}\right)$ if the charge is uniformly distributed over the surface, but is greater, $\left(\frac{3}{20\pi\epsilon_0}\frac{q^2}{R}\right)$, if the charge is uniformly distributed throughout the volume.

1.2 Induced charges:

If you hold a charge $+q$ near the uncharged conductor as shown in Figure: 3, the two will attract one another. The reason for this is that q will pull minus charges over to the near side and repel plus charges to the far side. (Another way to think of it is that the charge moves around in such a way as to cancel off the field of q for points inside the conductor, where the total field must be zero).

Since the negative induced charge is closer to q, there is a net force of attraction. By the way when we say the field, charge or potential "inside" a conductor, we mean in the "meat" of the conductor; if there is some **cavity** in the conductor, and within that cavity there is some charge, then **the field in the cavity will not be zero.**

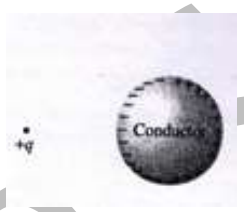

Figure: 3

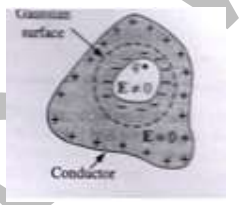

Figure: 4

But in a remarkable way the cavity and its contents are electrically isolated from the outside world by the surrounding conductors as shown in Figure: 4.

No external fields penetrate the conductor; they are canceled at the outer surface by the induced charge there. Similarly, the field due to the charges within the cavity is killed off, for all exterior points, by the induced charge on the inner surface. However, the compensating charge left over on the outer surface of the conductor effectively "communicates" the presence of q to the outside world. Incidentally, the total charge induced on the cavity wall is equal and opposite to the charge inside, for if we surround the cavity with a Gaussian surface, all points of which are in conductor as in Figure: 4.

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = 0$$

And hence (by Gauss's law) the net enclosed charge must be zero. But

$$Q_{enc} = q + q_{induced} = 0, \qquad \text{so } q_{induced} = -q.$$

If a cavity surrounded by conducting material is itself empty of charge, then the field within the cavity is zero. For any field line would have to begin and end on the cavity wall, going from a plus charge to minus charge see Figure: 5.

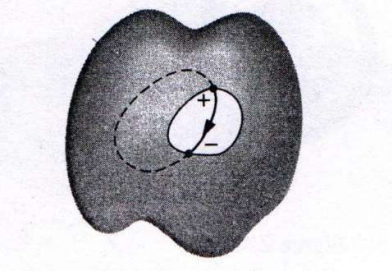

Figure: 5

Letting that field line be part of a closed loop, the rest of which is entirely inside the conductor (where $\boldsymbol{E} = 0$), the integral $\oint \boldsymbol{E} \cdot d\boldsymbol{l}$ is distinctly positive, [in violation of $\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$, the integral around a closed path is evidently zero]

It follows that $\boldsymbol{E} = 0$ within an empty cavity, and there is in fact no charge on the surface of the cavity. This is why you are relatively safe inside a metal car during thunderstorm----you may get cooked, if lightning strikes, but you will not be electrocuted. The same principle applies to the placement of sensitive apparatus inside a grounded **FARADAY CAGE**, to shield out stray electric fields. In practice, the enclosure doesn't even have to be solid conductor---chicken wire will often suffice. Working of Faraday's cage shown in Figure: 6, 7, 8.

Figure: 6

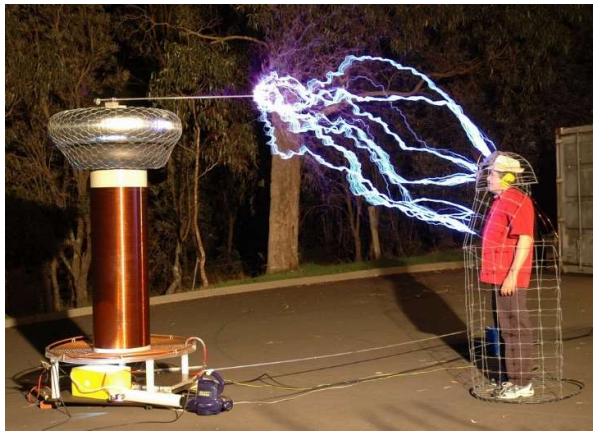

Figure: 7

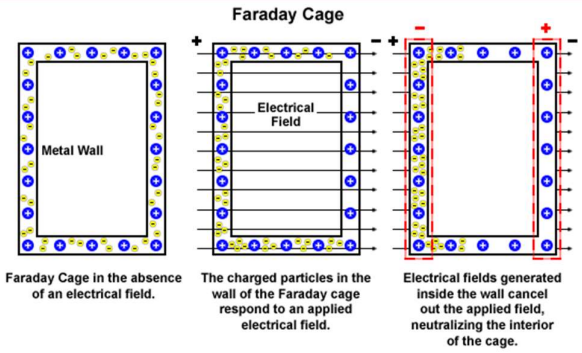

Figure: 8

Example: **1** An uncharged spherical conductor centered at the origin has a cavity of some weird shape carved out of it as shown in Figure: 9. Somewhere within the cavity is a charge q. What is the field outside the sphere?

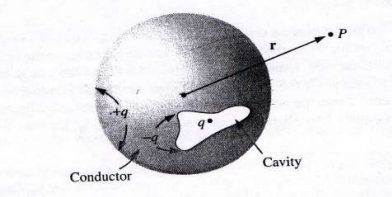

Figure: 9

Solution: We may think that the answer depends on the shape of the cavity and on the placement of the charge. But it is not so, the answer is as usual $\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\boldsymbol{r}}$.

In the cavity of the conductor, charge $+q$ induces an opposite charge $-q$ on the wall of the cavity, which distributes itself in such a way that its field cancels that of q, for all points exterior to the cavity. Since the conductor carries no net charge, this leaves $+q$ to distribute itself uniformly over the surface of the sphere. It's uniform because the asymmetrical influence of the point charge $+q$ is negated by that of the induced charge $-q$ on the inner surface. For points outside the sphere, then, the only thing that survives is the field of the leftover $+q$, uniformly distributed over the outer surface.

There are actually three fields at work here, $\boldsymbol{E}_q, \boldsymbol{E}_{induced}$ and $\boldsymbol{E}_{leftover}$. We know that the sum of the three is zero inside the conductor, it is possible that the first two alone cancel, while the third is separately zero there. There exists a way of distributing $-q$ over the inner surface so as to cancel the field of q at all exterior points. For that same cavity could have been carved out of a huge spherical conductor with a radius of 27 miles or light years or whatever. In that case the

leftover $+q$ on the outer surface is simply too far away to produce a significant field, and the other two fields would have to accomplish the cancellation by themselves. There is always distributing the charge on a conductor so as to make the field inside zero.

1.3 Surface charge and the force on a conductor:

We know, the field inside a conductor is zero. The field immediately outside is $\boldsymbol{E} = \frac{\sigma}{\epsilon_0}\hat{n}$. Here σ is surface charge, the field $\boldsymbol{E}$ is normal to the surface. In terms of potential one can say $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$ These equations help us to calculate the surface charge on a conductor.

In the presence of an electric field, a surface charge will, naturally, experience a force; the force per unit area, f is $\sigma\boldsymbol{E}$. But for the electric field is discontinuous at a surface charge, so which value are we supposed to use, $\boldsymbol{E}_{above}$ or $\boldsymbol{E}_{below}$ or something in between? The answer is that **we should use the average of the two.**

$$f = \sigma \boldsymbol{E}_{average} = \frac{1}{2}\sigma(\boldsymbol{E}_{above} + \boldsymbol{E}_{below})$$

To understand the reason, let's focus our attention on a small patch of surface surrounding the point in question as shown in the Figure: 10.

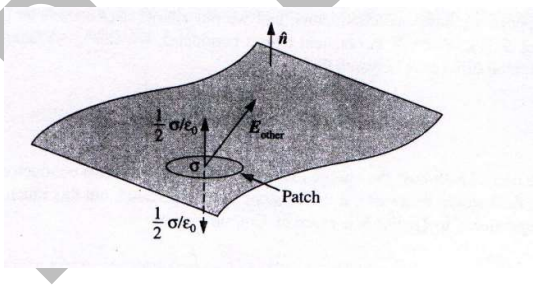

Figure: 10

Make it tiny enough so it is essentially flat and the surface charge on it is essentially constant. The total field consists of two parts ---that attributable to the patch itself, and that due to everything else (other regions of the surface, as well as any external sources that may be present.)

$$\boldsymbol{E} = \boldsymbol{E}_{pa} \quad + \boldsymbol{E}_{oth}$$

Now the patch cannot exert a force on itself, any more than you can lift yourself by standing in a basket and pulling up on the handles. The force on the patch, then, is due exclusively to $\boldsymbol{E}_{other}$,

and this suffers no discontinuity (if we remove the patch, the field in the "hole" would be perfectly smooth). The discontinuity is due entirely to the charge on the patch, which puts out a field $\frac{\sigma}{2\epsilon_0}$ on either side, pointing away from the surface, thus,

$$\boldsymbol{E}_{abov} = \boldsymbol{E}_{other} + \frac{\sigma}{2\epsilon_0}\hat{n}, \qquad \boldsymbol{E}_{below} = \boldsymbol{E}_{other} - \frac{\sigma}{2\epsilon_0}\hat{n}$$

and thus

$$\boldsymbol{E}_{other} = \frac{1}{2}(\boldsymbol{E}_{above} + \boldsymbol{E}_{below}) = \boldsymbol{E}_{average}$$

Averaging is really just a device for removing the contribution of the patch itself. That argument applies to any surface charge; in the particular case of a conductor, the field is zero inside and $\frac{\sigma}{\epsilon_0}\hat{n}$ outside, so the average is $\frac{\sigma}{2\epsilon_0}\hat{n}$. The force per unit area is $f = \frac{1}{2\epsilon_0}\sigma^2\hat{n}$. This amounts to an outward electrostatic pressure on the surface, tending to draw the conductor into the field, regardless of the sign of σ. Expressing the pressure in terms of the field just outside the surface, $P = \frac{\epsilon_0}{2}\,E^2$.

2. Laplace's Equation:

The primary task of electrostatics is to find the electric field of a given stationary charge distribution. In principle, this purpose is accomplished by Coulomb's law, written as

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{r^2}\,\hat{\boldsymbol{r}}\,d\tau' \quad (1)$$

Integrals of this type is bit difficult to calculate for any simplest charge configurations. We can do this by exploiting symmetry and using Gauss's law. But the best strategy is first to calculate potential V, which is in general given as

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{r}\,d\tau' \quad (2)$$

Still, even this integral is often too tough to handle analytically. Moreover, in problems involving conductors ρ itself may not be known in advance: since charge is free to move around, the only thing we control directly is the total charge (or perhaps the potential) of each conductor. In such cases it is fruitful to recast the problem in differential form, using Poisson's equation i.e.

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \quad (3)$$

which, together with appropriate boundary with conditions is equivalent to equation (2). Very often, in fact, we are interested in finding the potential in a region where $\rho = 0$. If $\rho = 0$ everywhere, of course, then $V = 0$, and there is nothing further to say---that's not what we mean. There may be plenty of charge elsewhere, but we are confining our attention to places where there is no charge. In such cases Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0 \quad (4)$$

Or in Cartesian coordinates,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad (5)$$

This formula is so fundamental to the subject that one might almost say electrostatics is the study of Laplace's equation. At the same time, it is a ubiquitous equation, appearing in such diverse branches of Physics as gravitation and magnetism, the theory of heat, and the study of soap bubbles. In mathematics it plays a major role in analytic function theory. To get a feel for Laplace's equation and its solutions (which are called **harmonic functions**), we shall begin with the one-two-dimension versions, which are easier to picture and illustrate all the essential properties of the three-dimensional case (though the one-dimensional example lacks the richness of the other two).

2.1 Laplace's Equation in One Dimension:

Suppose V depends on only one variable, x, Then Laplace's equation becomes

$$\frac{d^2 V}{dx^2} = 0$$

The general solution is

$$V(x) = mx + b$$

The equation for a straight line. It consists of two undetermined constants m and b, as is appropriate for a second-order (ordinary) differential equation. They are fixed, in any particular case by the boundary conditions of that problem. For instance, it might be specified that $V = 4$

at $x = 1$ and $V = 0$ at $x = 5$. In that case $m = -1$ and $b = 5$, so $V = -x + 5$ as shown in the Figure: 11.

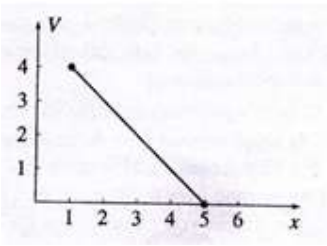

Figure: 11

Here we will notice two features of this result, they may seem silly and obvious in one dimension, where one can write down the general solution explicitly, but the analogs in two and three dimensions are powerful and by no means obvious:

1. $V(x)$ is the average of $V(x + a)$ and $V(x - a)$, for any a:

$$V(x) = \frac{1}{2}[V(x + a) + V(x - a)]$$

   Laplace's equation is a kind of averaging instruction; it tells us to assign to the point x the average of the values to the left and to the right of the x. Solutions to Laplace's equation are fit the end points properly.

2. Laplace's equation tolerates no local maxima or minima: extreme values of V must occur at the end points. Actually, this is the consequences of (1), for if there were a local maximum, V at the point would be greater than on either side, and therefore could not be average. (Ordinarily, you expect the second derivative to be a negative at a maximum and positive at a minimum. Since Laplace's equation requires, on the contrary, that the second derivative is zero, it seems reasonably that the solutions should exhibit no extrema. However, this is not a proof, since there exist functions that have maxima and minima at points where the second derivative vanishes: x^4, for example has such a minimum at the point at $x = 0$.)

2.2 Laplace's Equation in Two Dimensions:

If V depends on two variables, Laplace's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

This is **no longer an ordinary differential equation** (that is, one involving ordinary derivatives only): **it is a partial differential equation**. As a consequence, some of the simple rules you may

be familiar with do not apply. For instance, the general solution of this equation doesn't contain just two arbitrary constant – or, for that matter, any finite number – despite the fact that it's a second –order equation. Indeed, one cannot write down a general solution (at least, not in a closed form like in the case of first order i.e. $V(x) = mx + b$) . Nevertheless, it is possible to deduce certain properties common to all solutions.

It may help to have a physical example in mind. Picture a thin rubber sheet (or a soap film) stretched over some support. For definiteness, suppose you take a cardboard box, cut a wavy line all the way around, and remove the top part as shown in the Figure: 12.

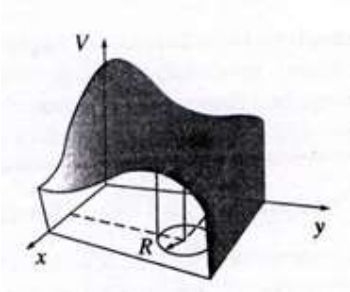

Figure: 12

Now glue a tightly stretched rubber membrane over the box, so that it fits like a drum head (it won't be a flat drum head, of course, unless you choose to cut the edges off straight). Now, if you lay out coordinates (x, y) on the bottom of the box, the height $V(x, y)$ of the sheet above the point (x, y) will satisfy Laplace's equation. (The one- dimensional analog would be a rubber band stretched between two points. Of course, it would be straight line.)

Harmonic functions in two dimensions have the same properties we noted in one dimension:

1. The value of V at a point (x, y) is the average of those around the point. More precisely, if you draw a circle of any radius R about that point (x, y), the average value of V on the circle is equal to the value at the center:

$$V(x, y) = \frac{1}{2\pi R} \oint V \, dl$$

(This, incidentally, suggests the **method of relaxation** on which computer solutions to Laplace's equation are based: Starting with specified values for V at the boundary, and reasonable guesses for V on a grid of interior points, the first pass reassigns to each point the average of its nearest neighbors. The second pass repeats the process, using the corrected values, and so on. After a few iterations, the numbers begin to settle down, so that subsequent

passes produce negligible changes, and a numerical solution to Laplace's equation, with the given boundary values, has been achieved)

2. V has **no local maxima or minima**; all extrema occur at the boundaries, (As before, this follows from (1).) Again, Laplace's equation picks the most featureless function possible, consistent with the **boundary conditions: no hills, no valleys**, just the smoothest surface available. For instance, if you put a ping-pong ball on the stretched rubber sheet as shown in Figure: 12, it will roll over to one side and fall off-it will not find a "pocket" somewhere to settle into, for Laplace's equation allows no such dents in the surface. From a geometrical point of view, just as a straight line is the shortest distance between two points, so a harmonic function in two dimensions minimizes the surface area spanning the given boundary line.

2.3 Laplace's equation in Three Dimensions:

Laplace's equation in three dimensions is written as:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

Harmonic functions in three dimensions have the same properties we noted as in one dimension and two dimensions:

1. The value of V at point $\boldsymbol{r}$ is the average value of V over a spherical surface of radius R centered at $\boldsymbol{r}$:

$$V(r)=\frac{1}{4\pi R^2}\oint V\,da$$

2. As a consequence, ***V* can have no local maxima or minima;** the extreme values of V must occur at the boundaries. (For if V had a local maximum at $\boldsymbol{r}$, then by the very nature of maximum I could draw a sphere around r over which all values of V – and a fortiori the average- would be less than at $\boldsymbol{r}$.)

   Proof: Let's begin by calculating the average potential over a spherical surface of radius R due to a single point charge q located outside the sphere. We may as well center the sphere at the origin and choose coordinates so that q lies on the $z-axis$ as shown in Figure: 13.

   The potential at a point on the surface is

$$V(r)=\frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

   Where:

$$r^2=z^2+R^2-2zR\cos\theta$$

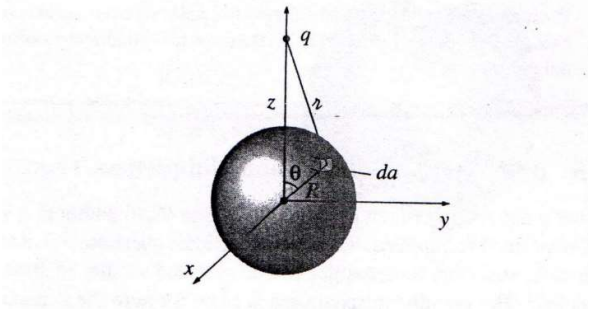

Figure: 13

So

$$r = [z^2 + R^2 - 2zR\cos\theta]^{1/2}$$

$$V_{ave} = \frac{1}{4\pi R^2}\int V(r)\, da$$

Here $da = R^2 \sin\theta \; d\theta \; d\emptyset$

$$V_{ave} = \frac{1}{4\pi R^2}\int \frac{q}{4\pi\epsilon_0 r} R^2 \sin\theta \; d\theta \; d\emptyset$$

$$V_{ave} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int r^{-1} R^2 \sin\theta \; d\theta \; d\emptyset$$

$$V_{ave} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int [z^2 + R^2 - 2zR\cos\theta]^{-1/2} R^2 \sin\theta \; d\theta \; d\emptyset$$

$$V_{ave} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int_0^{\pi} [z^2 + R^2 - 2zR\cos\theta]^{-1/2} R^2 \sin\theta \; d\theta \int_0^{2\pi} d\emptyset$$

$$V_{ave} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\, 2\pi R^2 \int_0^{\pi} [z^2 + R^2 - 2zR\cos\theta]^{-1/2} \sin\theta \; d\theta$$

Let

$$t = z^2 + R^2 - 2zR\cos\theta$$

$$dt = 2zR\sin\theta\, d\theta$$

$$as\ t = z^2 + R^2 - 2zR\cos\theta$$

$$\text{for } \theta = 0, \cos\theta = 1, t = z^2 + R^2 - 2zR = (z - R)^2$$

$$\text{for } \theta = \pi, \cos\theta = -1, t = z^2 + R^2 + 2zR = (z + R)^2$$

$$\therefore \sin\theta\, d\theta = \frac{1}{2zR}\, dt$$

$$V_{ave} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\, 2\pi R^2 \frac{1}{2zR}\int_{(z-R)^2}^{(z+R)^2} t^{-\frac{1}{2}} dt$$

$$V_{ave} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\, 2\pi R^2 \frac{1}{2zR}\frac{\left(t^{\frac{1}{2}}\right)_{(z-R)^2}^{(z+R)^2}}{1/2}$$

$$V_{ave} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\, 2\pi R^2 \frac{1}{2zR}\, 2\,[(z+R)-(z-R)]$$

$$= \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\, 2\pi R^2 \frac{1}{2zR}\, 2\,[z+R-z+R]$$

$$= \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\, 2\pi R^2 \frac{1}{2zR}\, 2\cdot 2R$$

$$V_{ave} = \frac{1}{4\pi\epsilon_0}\frac{q}{z}$$

But this is precisely the potential due to q at the center of the sphere! By the superposition principle, the same goes for any collection of charges outside the sphere: their average potential over the sphere is equal to the net potential they produce at the center.

2.4 Separation of Variables:

Now we use the method of separation of variables to solve Laplace's equation directly, which is the physicist's favorite tool for solving partial differential equations. The method is applicable in circumstances where the potential (V) or the charge density (σ) is specified on the boundaries of some region, and we are asked to find the potential in the interior. The basic strategy is very simple: We look for solutions that are products of functions, each of which depends on only one of the coordinates. The algebraic details, however, can be formidable, so we are going to develop the method through a sequence of examples. We will start with Cartesian coordinates and then do spherical coordinates.

2.4.1 Cartesian Coordinates:

Example: 2 Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$ as shown in the Figure: 14. The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates and maintained at a specific potential $V_0(y)$. Find the potential inside the slot.

Solution: The configuration is independent of z, so this is really a two-dimensional problem. In mathematical term, we must solve Laplace's equation.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad (1)$$

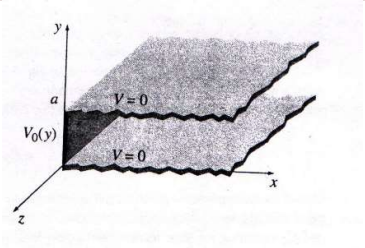

Figure: 14

Subject to the boundary conditions:

(i) $V = 0\ when\ y = 0,$ $(2a)$

(ii) $V = 0\ when\ y = a,$ $(2b)$

(iii) $V = V_0(y)\ when\ x = 0,$ $(2c)$

(iv) $V \to 0\ as\ x \to \infty$ $(2d)$

(The latter, although nor explicitly stated in the problem, is necessary on physical ground: as you get farther and farther away from the hot strip at $x = 0$, the potential should drop to zero.) Since the potential is specified on all boundaries, the answer is uniquely determined.

We use method of separation of variables to solve Laplace's equation. we assume that the potential $V(x, y)$ is written as

$$V(x, y) = X\,(x)Y(y) \quad (3)$$

Here $X(x)$ is a purely x dependent part and $Y(y)$ is a purely y dependent part of the potential $V(x, y)$.

Now substitute value of equation (3) in equation (1), we obtain

$$\frac{\partial^2 X\,Y}{\partial x^2} + \frac{\partial^2 X\,Y}{\partial y^2} = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

Divide above equation with XY, we obtain

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0 \quad (4)$$

Here the first term depends only on x and the second only on y. Let

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = C_1, \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = C_2, \qquad with\ C_1 + C_2 = 0 \quad (5)$$

One of these constants is positive and the other negative (or perhaps both are zero). But here we take C_1 positive and C_2 negative. Let

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2, \qquad \therefore \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2$$

$$\frac{\partial^2 X}{\partial x^2} = k^2 X \quad (6)$$

$$\frac{\partial^2 Y}{\partial y^2} = -k^2 Y \quad (7)$$

Solutions of equations (6) and (7)

$$X(x) = Ae^{kx} + Be^{-kx} \quad (8)$$

$$Y(y) = C\sin ky + D\cos ky \quad (9)$$

Here A, B, C, D are undetermined constants.

Therefore, by substituting values of equations (8), (9) in equation (3), it becomes

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky) \quad (10)$$

By applying boundary condition (iv): equation (2d): $V \to 0\ as\ x \to \infty$, equation (10) LHS is zero. Therefore, RHS must be zero. It is possible only if A is zero.

$$A = 0 \quad (11)$$

Therefore, equation (10) becomes

$$V(x, y) = Be^{-k}\ (C\sin ky + D\cos ky)$$

By absorbing B into C and D

$$V(x, y) = e^{-kx}(C\sin ky + D\cos ky) \quad (12)$$

Now applying boundary condition (i): equation (2a): $V = 0\ when\ y = 0$, in equation (12), LHS is zero, therefore, RHS must be zero

$$0 = e^{-kx}(C \sin 0 + D \cos 0)$$

$$0 = C \sin 0 + D \cos 0$$

$$D = 0 \quad (13)$$

Therefore, equation (12) becomes

$$V(x, y) = Ce^{-kx} \sin ky \quad (14)$$

By applying boundary condition (ii): equation (2b): $V = 0 \; when \; y = a$, in equation (14), we have

$$0 = Ce^{-kx} \sin ka$$

$$\sin ka = 0$$

It is possible only: $ka = n\pi$ or

$$k = \frac{n\pi}{a}, (n = 1, 2, 3, \dots) \quad (15)$$

Therefore, equation (14) becomes

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right) \quad (16)$$

Now boundary condition (iii): equation (2c): $V = V_0(y) \; when \; x = 0$, equation (16) becomes

$$V(0, y) = V_0(y) = \sum_{n=1}^{\infty} C_n e^0 \sin\left(\frac{n\pi y}{a}\right)$$

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y) \quad (17)$$

This sum is a Fourier sine series. And Dirichlet's theorem guarantees that virtually any function $V_0(y)$ (it can even have a finite number of discontinuities) can be expanded in such a series. To obtain value of C_n, multiply equation (17) by $\sin\left(\frac{n'\pi y}{a}\right)$, where n' is a positive integer and integrate from 0 to a we obtain:

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy \quad (18)$$

In LHS of equation (18):

for $n' \neq n$:

we know that $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

$$\therefore \int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \frac{1}{2}\int_0^a \left[\cos(n'-n)\frac{\pi y}{a} - \cos(n'+n)\frac{\pi y}{a}\right]dy$$

$$= \frac{1}{2}\left[\frac{\sin(n'-n)\frac{\pi y}{a}}{(n'-n)\frac{\pi}{a}} - \frac{\sin(n'+n)\frac{\pi y}{a}}{(n'+n)\frac{\pi}{a}}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(n'-n)\frac{\pi a}{a} - \sin 0}{(n'-n)\frac{\pi}{a}} - \frac{\sin(n'+n)\frac{\pi a}{a} - \sin 0}{(n'+n)\frac{\pi}{a}}\right]$$

$$= \frac{1}{2}\left[\frac{\sin(n'-n)\pi - \sin 0}{(n'-n)\frac{\pi}{a}} - \frac{\sin(n'+n)\pi - \sin 0}{(n'+n)\frac{\pi}{a}}\right] = 0$$

$$\because \sin n\pi = 0\ (n = 0, \pm1, \pm2, \dots)$$

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = 0 \quad for\ n' \neq n \quad (19)$$

for $n' = n$, we have

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \int_0^a \sin^2\left(\frac{n\pi y}{a}\right)dy$$

$$= \frac{1}{2}\int_0^a \left[1 - \cos\left(\frac{2n\pi y}{a}\right)\right]dy$$

$$= \frac{1}{2}\left[y - \frac{\sin\left(\frac{2n\pi y}{a}\right)}{\left(\frac{2n\pi}{a}\right)}\right]_0^a$$

$$= \frac{1}{2}\left[a - 0 - \frac{\sin\left(\frac{2n\pi y}{a}\right) - \sin 0}{\left(\frac{2n\pi}{a}\right)}\right] = \frac{a}{2}$$

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \frac{a}{2} \quad for\ n' = n \qquad (20)$$

Therefore,

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \begin{cases} 0, & for\ n' \neq n \\ \frac{a}{2}, & for\ n' = n \end{cases} \qquad (21)$$

Therefore, equation (18) now becomes:

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)dy \qquad (18)$$

$$C_n \frac{a}{2} = \int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy \qquad (22)$$

Suppose the strip at $x = 0$ is a metal plate with constant potential V_0, then equation (22) becomes:

$$C_n = \frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right)dy = \frac{2V_0}{a}\left[-\frac{\cos\left(\frac{n\pi y}{a}\right)}{\frac{n\pi}{a}}\right]_0^a$$

$$= -\frac{2V_0}{n\pi}\left[\cos\left(\frac{n\pi a}{a}\right) - \cos 0\right]$$

$$C_n = \frac{2V_0}{n\pi}[1 - \cos n\pi] \qquad (23)$$

We know that:

$$\cos n\pi = \begin{cases} -1, & for\ n = 1, 3, 5 \ldots (odd\ numbers) \\ 1, & for\ n = 2, 4, 6\ \ldots (even\ numbers) \end{cases}$$

Therefore, equation (23) becomes

$$C_n = \frac{2V_0}{n\pi}[1 - \cos n\pi] = \begin{cases} 0, & if\ n\ is\ even \\ \frac{4V_0}{n\pi}, & if\ n\ is\ odd \end{cases} \qquad (24)$$

Therefore, equation (16) now becomes:

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right) \quad (16)$$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right) \quad (25)$$

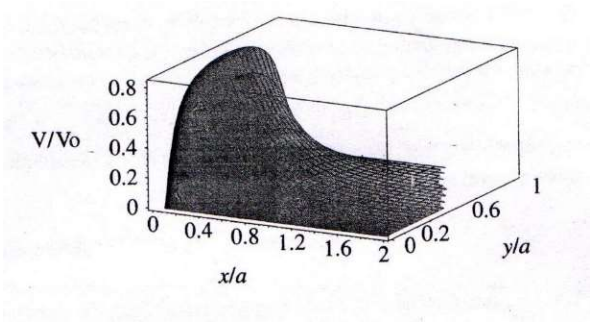

Figure: 15

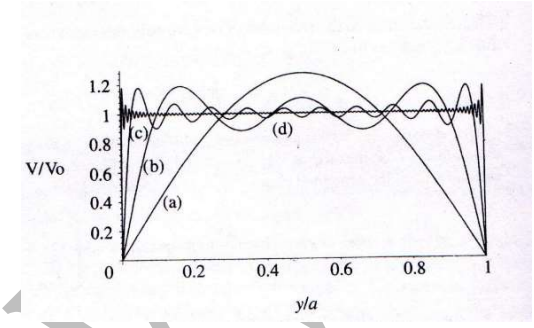

Figure: 16

Figure: 15 is a plot of this potential; Figure: 16 shows how the first few terms in the Fourier series combine to make a better and better approximation to the constant V_0 : (a) is $n = 1$ only, (b) includes n upto 5, (c) is the sum of the first 10 terms, and (d) is the sum of the first 100 terms. Incidentally, the infinite series in equation (25) can be summed explicitly; the result is

$$V(x,y) = \frac{2V_0}{\pi} \tan^{-1}\left(\frac{\sin\left(\frac{\pi y}{a}\right)}{\sinh\left(\frac{\pi x}{a}\right)}\right) \quad (26)$$

In above solution, Laplace's equation is obeyed and the four boundary conditions of equation (2) are satisfied.

Example: 3 Two infinitely grounded metal plates, at $y = 0$ and $y = a$. are connected at $x = \pm b$ by metal strips maintained at a constant potential V_0, as shown in the Figure: 17. (a thin layer of insulation at each corner prevents them from shorting out). Find the potential inside the resulting rectangular pipe.

Solution: Once again the configuration is independent of z. Our problem is to solve Laplace's equation.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad (1)$$

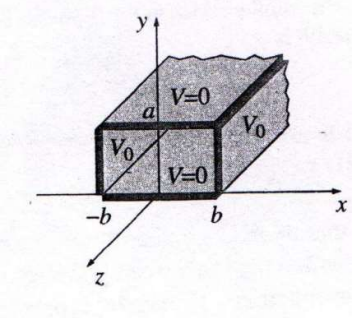

Figure: 17

Subject to the boundary conditions:

(i) $V = 0\ when\ y = 0,$ $(2a)$

(ii) $V = 0\ when\ y = a,$ $(2b)$

(iii) $V = V_0\ when\ x = b,$ $(2c)$

(iv) $V = V_0\ when\ x = -b$ $(2d)$

We use method of separation of variables to solve Laplace's equation. we assume that the potential $V(x, y)$ is written as

$$V(x, y) = X\ (x)Y(y) \quad (3)$$

Here $X(x)$ is a purely x dependent part and $Y(y)$ is a purely y dependent part of the potential $V(x, y)$.

Now substitute value of equation (3) in equation (1), we obtain

$$\frac{\partial^2 X\,Y}{\partial x^2} + \frac{\partial^2 X\,Y}{\partial y^2} = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

Divide above equation with XY, we obtain

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0 \quad (4)$$

Here the first term depends only on x and the second only on y. Let

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = C_1, \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = C_2, \qquad with\ C_1 + C_2 = 0 \quad (5)$$

One of these constants is positive and the other negative (or perhaps both are zero). But here we take C_1 positive and C_2 negative. Let

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2, \qquad \therefore \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2$$

$$\frac{\partial^2 X}{\partial x^2} = k^2 X \quad (6)$$

$$\frac{\partial^2 Y}{\partial y^2} = -k^2 Y \quad (7)$$

Solutions of equations (6) and (7)

$$X(x) = Ae^{kx} + Be^{-k} \quad (8)$$

$$Y(y) = C\sin ky + D\cos ky \quad (9)$$

Here A, B, C, D are undetermined constants.

Therefore, by substituting values of equations (8), (9) in equation (3), it becomes

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky) \quad (10)$$

This time, we cannot set $A = 0$; as the region is between $\pm b$, does not extend to $x = \infty$. So e^{kx} is perfectly acceptable. Here the situation is symmetric (about origin) with respect to x. So

$$V(-x,y) = V(x,y) \quad (11)$$

Which is possible only if

$$A = B \quad (12)$$

We know that:

$$e^{kx} + e^{-kx} = 2\cosh kx \quad (13)$$

By substituting values of equation (12) and (13) in equation (10), we have

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky) \quad (10)$$

$$V(x,y) = A(e^{kx} + e^{-kx})(C\sin ky + D\cos ky)$$

$$V(x,y) = 2A\cosh kx\ (C\sin ky + D\cos ky) \quad (14)$$

By absorbing $2A$ into C and D:

$$V(x,y) = \cosh kx\ (C\sin ky + D\cos ky) \quad (15)$$

Now applying boundary condition (i): equation (2a): $V = 0\ when\ y = 0$, in equation (15), LHS is zero, therefore, RHS must be zero

$$0 = \cosh kx\ (C\sin 0 + D\cos 0)$$

$$0 = C\sin 0 + D\cos 0$$

$$D = 0 \quad (16)$$

Therefore, equation (15) becomes

$$V(x,y) = C\cosh kx\sin ky \quad (17)$$

By applying boundary condition (ii): equation (2b): $V = 0\ when\ y = a$, in equation (17), we have

$$0 = C\cosh kx\sin ka$$

$$\sin ka = 0$$

It is possible only: $ka = n\pi$ or

$$k = \frac{n\pi}{a}, (n = 1, 2, 3, \dots) \quad (18)$$

Boundary conditions (i) and (ii) require as before that $D = 0$ and $k = n\pi/a$, so

$$V(x,y) = C\cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \quad (19)$$

Because $V(x,y)$ is even in x, it will automatically meet condition (iv) if it fits (iii). It remains, therefore, to construct the general linear combination,

$$V(x,y) = \sum_{n=1}^{\infty} C_n\cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \quad (20)$$

Here, coefficients C_n in such a way as to satisfy condition (iii):

$$V(b,y) = \sum_{n=1}^{\infty} C_n\cosh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right) = V_0 \quad (21)$$

coefficients C_n are:

To obtain value of C_n, multiply equation (21) by $\sin\left(\frac{n'\pi y}{a}\right)$, where n' is a positive integer and integrate from 0 to a we obtain:

$$\sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right) \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \int_0^a V_0 \sin\left(\frac{n'\pi y}{a}\right) dy \quad (22)$$

In LHS of equation (22):

for $n' \neq n$: we know that $2 \sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$

$$\therefore \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \frac{1}{2} \int_0^a \left[\cos(n' - n)\frac{\pi y}{a} - \cos(n' + n)\frac{\pi y}{a}\right] dy$$

$$= \frac{1}{2}\left[\frac{\sin(n' - n)\frac{\pi y}{a}}{(n' - n)\frac{\pi}{a}} - \frac{\sin(n' + n)\frac{\pi y}{a}}{(n' + n)\frac{\pi}{a}}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(n' - n)\frac{\pi a}{a} - \sin 0}{(n' - n)\frac{\pi}{a}} - \frac{\sin(n' + n)\frac{\pi a}{a} - \sin 0}{(n' + n)\frac{\pi}{a}}\right]$$

$$= \frac{1}{2}\left[\frac{\sin(n' - n)\pi - \sin 0}{(n' - n)\frac{\pi}{a}} - \frac{\sin(n' + n)\pi - \sin 0}{(n' + n)\frac{\pi}{a}}\right] = 0$$

$$\because \sin n\pi = 0 \ (n = 0, \pm 1, \pm 2, \dots)$$

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = 0 \quad for\ n' \neq n \quad (23)$$

$for\ n' = n$, we have

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \int_0^a \sin^2\left(\frac{n\pi y}{a}\right) dy$$

$$= \frac{1}{2}\int_0^a \left[1 - \cos\left(\frac{2n\pi y}{a}\right)\right] dy$$

$$= \frac{1}{2}\left[y - \frac{\sin\left(\frac{2n\pi y}{a}\right)}{\left(\frac{2n\pi}{a}\right)} \right]_0^a$$

$$= \frac{1}{2}\left[a - 0 - \frac{\sin\left(\frac{2n\pi y}{a}\right) - \sin 0}{\left(\frac{2n\pi}{a}\right)} \right] = \frac{a}{2}$$

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \frac{a}{2} \quad for\ n' = n \qquad (24)$$

Therefore,

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \begin{cases} 0, & for\ n' \neq n \\ \frac{a}{2}, & for\ n' = n \end{cases} \qquad (25)$$

Therefore, equation (22) now becomes:

$$\sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right) \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \int_0^a V_0 \sin\left(\frac{n'\pi y}{a}\right) dy \qquad (22)$$

$$C_n \cosh\left(\frac{n\pi b}{a}\right) \frac{a}{2} = \int_0^a V_0 \sin\left(\frac{n\pi y}{a}\right) dy$$

$$C_n \cosh\left(\frac{n\pi b}{a}\right) = \frac{2V_0}{a} \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy \qquad (26)$$

$$C_n \cosh\left(\frac{n\pi b}{a}\right) = \frac{2V_0}{a} \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy = \frac{2V_0}{a}\left[-\frac{\cos\left(\frac{n\pi y}{a}\right)}{\frac{n\pi}{a}} \right]_0^a$$

$$= -\frac{2V_0}{n\pi}\left[\cos\left(\frac{n\pi a}{a}\right) - \cos 0 \right]$$

$$C_n \cosh\left(\frac{n\pi b}{a}\right) = \frac{2V_0}{n\pi}[1 - \cos n\pi] \qquad (27)$$

We know that:

$$\cos n\pi = \begin{cases} -1, & for\ n = 1, 3, 5 \ldots (odd\ numbers) \\ 1, & for\ n = 2, 4, 6 \ \ldots (even\ numbers) \end{cases}$$

Therefore, equation (23) becomes

$$C_n \cosh\left(\frac{n\pi b}{a}\right) = \frac{2V_0}{n\pi}[1 - \cos n\pi] = \begin{cases} 0, & if\ n\ is\ even \\ \dfrac{4V_0}{n\pi}, & if\ n\ is\ odd \end{cases} \quad (28)$$

$$C_n \cosh\left(\frac{n\pi b}{a}\right) = \begin{cases} 0, & if\ n\ is\ even \\ \dfrac{4V_0}{n\pi}, & if\ n\ is\ odd \end{cases} \quad (29)$$

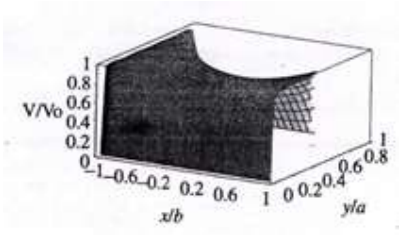

Figure: 18

The potential in this case is given by

$$V(x,y) = \frac{4V_0}{n\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} \frac{\cosh\left(\frac{n\pi x}{a}\right)}{\cosh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi y}{a}\right) \quad (30)$$

Example: 4 An infinitely long rectangular metal strip (side a and b) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y,z)$, as indicated is Figure: 19. Find the potential inside the pipe.

Solution: This is a three-dimensional problem,

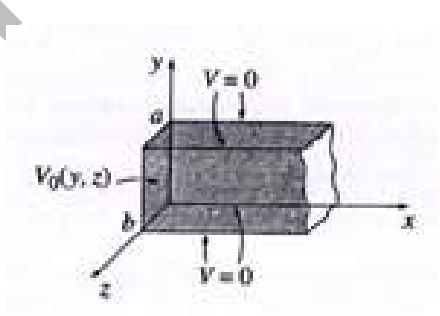

Figure: 19

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad (1)$$

Subject to the boundary conditions

(i) $V = 0\ when\ y = 0,$ $(2a)$

(ii) $V = 0\ when\ y = a,$ $(2b)$

(iii) $V = 0\ when\ z = 0,$ $(2c)$

(iv) $V = 0\ when\ z = b,$ $(2d)$

(v) $V \to 0\ as\ x \to \infty,$ $(2e)$

(vi) $V = V_0(y, z)\ when\ x = 0$ $(2f)$

We use method of separation of variables to solve Laplace's equation. we assume that the potential $V(x, y, z)$ is written as:

$$V(x, y, z) = X(x)Y(y)Z(z) \quad (3)$$

Here $X(x)$ is a purely x dependent part $Y(y)$ is a purely y dependent part and $Z(z)$ is a purely z dependent part of the potential $V(x, y, z)$.

Now substitute value of equation (3) in equation (1), we obtain

$$\frac{\partial^2 X\,YZ}{\partial x^2} + \frac{\partial^2 X\,YZ}{\partial y^2} + \frac{\partial^2 X\,YZ}{\partial z^2} = 0$$

$$YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} = 0$$

Divide above equation with XYZ, we obtain

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0 \quad (4)$$

Here the first term depends only on x, second only on y and third term only on z. Let

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = C_1, \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = C_2, \qquad \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = C_3 \ \ with\ C_1 + C_2 + C_3 = 0 \quad (5)$$

One of these constants C_1 must be positive and C_2 and C_3 negative.

Suppose

$$C_2 = -k^2, \qquad C_3 = -l^2, \ \ \therefore C_1 = k^2 + l^2\ (As\ C_1 + C_2 + C_3 = 0\ ) \quad (6)$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2 + l^2, \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2, \qquad \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -l^2$$

$$\frac{\partial^2 X}{\partial x^2} = (k^2 + l^2)X \qquad (7)$$

$$\frac{\partial^2 Y}{\partial y^2} = -k^2 Y \qquad (8)$$

$$\frac{\partial^2 Z}{\partial z^2} = -l^2 Z \qquad (9)$$

Solutions of equations (7), (8) and (9)

$$X(x) = Ae^{\sqrt{k^2+l^2}x} + Be^{-\sqrt{k^2+l^2}x} \qquad (10)$$

$$Y(y) = C \sin ky + D \cos ky \qquad (11)$$

$$Z(z) = E \sin lz + F \cos lz \qquad (12)$$

Here A, B, C, D, E, F are undetermined constants. Boundary conditions (v) implies $A = 0$, (i) gives $D = 0$ and (iii) gives $F = 0$, where as (ii) and (iv) require that $k = n\pi/a$ and $l = m\pi/b$, where n and m are positive integer. Combining the remaining constants, we are left with

$$V(x, y, z) = C\, e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi z}{b}\right) \qquad (13)$$

This solution meets all the boundary conditions except (vi). It contains two unspecified integers (n and m). The most general linear combination is a double sum:

$$V(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\, e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi z}{b}\right) \qquad (14)$$

Equation (14) fits the remaining boundary condition: (vi): equation (2g),

$$V(0, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi z}{b}\right) = V_0(y, z) \qquad (15)$$

By approximate choice of the coefficients $C_{n,m}$.

To determine these constants, we multiply by $\sin(n'\pi y/a)\sin(m'\pi z/b)$. Where n' and m' are arbitrary positive integers, and integrate between 0 to a and 0 to b:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy \int_0^b \sin\left(\frac{m\pi z}{b}\right)\sin\left(\frac{m'\pi z}{b}\right)dz$$

$$= \int_0^a \int_0^b V_0(y,z)\sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi z}{b}\right)dy\,dz \qquad (16)$$

In the LHS of above equation:

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \frac{a}{2}, \qquad \int_0^b \sin\left(\frac{m\pi z}{b}\right)\sin\left(\frac{m'\pi z}{b}\right)dz = \frac{b}{2} \qquad (17)$$

Therefore, LHS of equation (16) is: $C_{n',m'}\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)$

$$C_{n,m}\frac{ab}{4} = \int_0^a \int_0^b V_0(y,z)\sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi z}{b}\right)dy\,dz$$

$$\therefore\ C_{n,m} = \frac{4}{ab}\int_0^a \int_0^b V_0(y,z)\sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi z}{b}\right)dy\,dz \qquad (18)$$

Equation (14) with equation (18) is the solution.

For instant if the end of the tube is a conductor at constant potential V_0. Equation (18) becomes:

$$C_{n,m} = \frac{4V_0}{ab}\int_0^a \int_0^b \sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi z}{b}\right)dy\,dz$$

$$C_{n,m} = \frac{4V_0}{ab}\int_0^a \sin\left(\frac{n'\pi y}{a}\right)dy \int_0^b \sin\left(\frac{m'\pi z}{b}\right)dz$$

$$C_{n,m} = \begin{cases} 0, & if\ n\ or\ m\ is\ even \\ \dfrac{16V_0}{\pi^2}, & if\ n\ and\ m\ are\ odd \end{cases} \qquad (19)$$

In this case

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n,m=1,3,5\ldots}^{\infty}\frac{1}{nm}e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right) \qquad (20)$$

2.4.2 Spherical Coordinates:

For objects having 'plane' boundaries, cartesian coordinates are very appropriate. But for round objects spherical coordinates are more natural or more useful.

Laplace's equation in spherical coordinates is written as:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\emptyset^2}=0 \quad (1)$$

Let the problem has azimuthal symmetry. So that V is independent of $\emptyset$. Therefore, equation (1) becomes:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0 \quad (2)$$

By using the method of separation of variables,

$$V(r,\theta)=R(r)\Theta(\theta) \quad (3)$$

Substituting value of equation (3) in equation (2), we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R\Theta}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial R\Theta}{\partial\theta}\right)=0$$

$$\frac{\Theta}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0$$

$$\Theta\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0$$

Dividing above equation with $V=R\Theta$,

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0 \quad (4)$$

In equation (4), the first term depends only on r, and the second only on θ, it follows that each term must be a constant.

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1) \quad (5)$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1) \quad (6)$$

Here $l(l+1)$ is a constant.

From equation (5), LHS: the radial equation is written as

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R \quad (7)$$

It has the general solution:

$$R(r) = Ar^l + \frac{B}{r^{l+1}} \quad (8)$$

Here A and B are arbitrary constants.

From equation (5), RHS: the angular equation is written as:

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\,\Theta \quad (9)$$

The solutions are Legendre's polynomials in the variable $\cos\theta$:

$$\Theta(\theta) = P_l(\cos\theta) \quad (10)$$

$P_l(x)$ is defined by the Rodrigues formula:

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \quad (11)$$

The first few Legendre polynomials are:

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{3x^2-1}{2} \\ P_3(x) &= \frac{5x^3-3x}{2} \\ P_4(x) &= \frac{35x^4-30x^2+3}{8} \\ P_5(x) &= \frac{63x^5-70x^3+15x}{8} \end{aligned}\right\} \quad (12)$$

Here $P_l(x)$ is an l^{th} order polynomial in x. It contains even powers, if l is even; and odd powers, if l is odd. The factor $\frac{1}{2^l l!}$ in $P_l(x)$ was chosen in order that

$$P_l(1) = 1 \quad (13)$$

The Rodrigues formula works only for nonnegative integer values of l. It provides us only one solution. But equation (9) is second order and it should possess two independent solutions, for every value of l. It turns out that these "other solutions" blow up at $\theta = 0$ and / or $\theta = \pi$, and are therefore, unacceptable on physical ground. For instance, the second solution for $l = 0$ is

$$\Theta(\theta) = \ln\left(tan\frac{\theta}{2}\right) \quad (14)$$

In the case of azimuthal symmetry, the most general separable solution to Laplace's equation, consistent with minimal physical requirements, is

$$V(r,\theta) = \left(Ar^l + \frac{B}{r^{l+1}}\right) P_l(cos\,\theta) \quad (15)$$

As, separation of variables yields in infinite set of solutions, one for each l. The general solution is the linear combination of separable solutions:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(cos\,\theta) \quad (16)$$

Example: 5 The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of raius R. Find the potential inside the sphere.

Solution: In this case $B_l = 0$ for all l, otherwise the potential would blow up at the origin. Thus

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l\, P_l(cos\,\theta) \quad (1)$$

At $r = R$ this must match the specified function $V_0(\theta)$:

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l\, P_l(cos\,\theta) = V_0(\theta) \quad (2)$$

This equation can be satisfied for an appropriate choice of coefficients A_l .The Legendre polynomials constitute a complete set of functions, on the interval $-1 \le x \le 1$ $(0 \le \theta \le \pi)$. We determine the constants by Fourier's trick. The Legendre polynomials are orthogonal functions.

$$\int_{-1}^{1} P_l(x) P_{l'}(x) dx = \int_0^{\pi} P_l(cos\,\theta) P_{l'}(cos\,\theta)\, sin\,\theta\, d\theta \quad (3)$$

$$= \begin{cases} 0, & if \quad l' \neq l \\ \dfrac{2}{2l+1}, & if \quad l' = l \end{cases} \quad (4)$$

Thus, multiplying equation (2) by $P_{l'}(cos\,\theta)\ sin\,\theta$ and integrating, we have

$$A_{l'} R^{l'} \frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta)\, P_{l'}(cos\,\theta)\, sin\,\theta\, d\theta$$

or

$$A_l = \frac{2l+1}{2R_l} \int_0^{\pi} V_0(\theta)\, P_l(cos\,\theta)\, sin\,\theta\, d\theta \quad (5)$$

Equation (1) is the solution to our problem, with the coefficients given by equation (5).

For instance, suppose we are told that the potential on the sphere is

$$V_0(\theta) = K\, sin^2(\theta/2) \quad (6)$$

Where k is constant. Using half angle formula, we write this as

$$V_0(\theta) = \frac{k}{2}(1 - cos\,\theta) = \frac{k}{2}\{P_0(cos\,\theta) - P_1(cos\,\theta)\} \quad (7)$$

Putting this into equation (2), we read off immediately that $A_0 = k/2,\ A_1 = -k/(2R)$ and all other A_l's vanish. Evidently,

$$V_0(r,\ \theta) = \frac{k}{2}\left[(r^0 P_0(cos\,\theta) - \frac{r^1}{R} P_1(cos\,\theta)\right] = \frac{k}{2}\left(1 - \frac{r}{R} cos\,\theta\right) \quad (8)$$

Example: 6 The potential $V_0(\theta)$ is again specified on the surface of a sphere of radius R, but this time we are asked to find the potential outside, assuming there is no charge there.

Solution: In this case it's the A_l's that must be zero (or else V would not go to zero at ∞), so

$$V_0(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(cos\,\theta) \quad (19)$$

At the surface of the sphere, we require that

$$V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(cos\,\theta) = V_0(\theta)$$

Multiplying by $P_l(cos\,\theta)\sin\theta$ and integrating-exploiting, again, the orthogonality relation Eq.15 –we have

$$\frac{B_{l'}}{R^{l'+1}}\frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta) P_{l'}(cos\,\theta)\, sin\,\theta\, d\theta$$

$$or\ B_l = \frac{2l'+1}{2} R^{l+1} \int_0^{\pi} V_0(\theta) P_l(cos\,\theta)\, sin\,\theta\, d\theta \quad (20)$$

This equation (19), with the coefficients given in equation (20), is the solution to our problem.

Example: 7 An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $E = E_0\hat{z}$. [The field will push positive charge to the "northern" surface of the sphere, leaving a negative charge on the 'southern' surface as in Figure: 20. This induced charge, in turn, distorts the field in the neighborhood of the sphere] Find the potential in the region outside the sphere.

Solution:

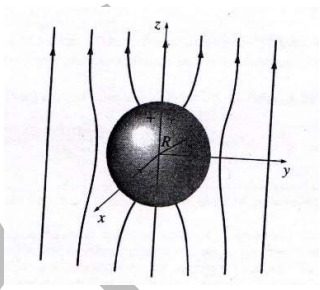

Figure: 20

The sphere is an equipotential ---we may as well set it to zero. Then by symmetry the entire xy plane is at potential zero. This time, however, V does not go to zero at large z. In fact, far from the sphere the field is $E = E_0\hat{z}$ and hence

$$V \to -E_0 z + C$$

Since $V = 0$ in the equipotential plane, the constant C must be zero. Accordingly, the boundary conditions for this problem are

(i) $V = 0\ for\ r < R$

(ii) $V \to -E_0 r\, cos\,\theta \quad for\ r \gg R$

We must fit these boundary conditions with a function of the form:

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(cos\,\theta)$$

The first condition yields

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0$$

Or

$$B_l = -A_l R^{2l+1}$$

So

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(cos\,\theta)$$

For $r \gg R$, the second term in parentheses is negligible, and therefore, condition (ii) requires that

$$\sum_{l=0}^{\infty} A_l\, r^l P_l(cos\,\theta) = -E_0 r\, cos\,\theta$$

Evidently, only one term is present: $l = 1$. In fact, since $P_l(cos\,\theta) = cos\,\theta$, we can read off immediately $A_l = -E_0$, all other A_l's zero.

Conclusion:

$$V(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) cos\,\theta$$

The first term $-E_0 r\, cos\,\theta$ is due to the external field, the contribution attributable to the induced charge is evidently

$$E_0 \frac{R^3}{r^2} cos\,\theta$$

If you want to know the induced charge density, it can be calculated in the usual way

$$\sigma(\theta) = -\epsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} = \epsilon_0 E_0 \left(1 + \frac{R^3}{r^3} \right) cos\,\theta \bigg|_{r=R}$$

3. Electric Fields in Matter:

3.1 Dielectrics:

In dielectrics all charges are attached to specific atoms or molecules. They are on a tight leash and they can do is move a bit within the atom or molecule. Such microscopic displacements are not as dramatic as the wholesale rearrangement of charge in a conductor, but their cumulative effects account for the characteristic behavior of dielectric materials. There are actually two principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule. They are (i) stretching and (ii) rotating.

3.2 Induced Dipole:

When a neutral atom when it is placed in an electric field $\boldsymbol{E}$, the atom as a whole is electrically neutral, there is a positively charged core (the nucleus) and negatively charged electron cloud surrounding it. These two regions of charge within the atom are influenced by the field: the nucleus is pushed in the direction of the field, and the electrons the opposite way. In principle, if the field is large enough, it can pull the atom apart completely "ionizing" it (the substance then becomes a conductor). With less extreme fields, however, an equilibrium is soon established, for if the center of the electron cloud does not coincide with the nucleus, these positive and negative charges attract one another, and this holds the atoms together.

The two opposing forces – $\boldsymbol{E}$ pulling the electrons and nucleus apart, their mutual attraction drawing them together-reach a balance, leaving the atom polarized, with plus charge shifted slightly one way, and minus the other. The atom now has a tiny dipole moment $\boldsymbol{p}$, which points in the same direction as $\boldsymbol{E}$. This induced dipole moment is approximately proportional to the $\boldsymbol{E}$.

$$\boldsymbol{p} = \boldsymbol{\alpha E} \quad (1)$$

Where α is the proportionality constant and is called **atomic polarizability**. Its value depends on the detailed structure of the particular atom. There is a list of some experimentally determined atomic polarizability.

Atomic polarizability ($\alpha/4\pi\epsilon_0$, in units of $10^{-3}\ \ m^3$)

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.76	0.396	24.1	1.64	43.4	59.6

For molecules the situation is not quite so simple, because frequently they polarize more readily in some directions than others. Carbon dioxide as shown in Figure: 21. For instance, has a polarizability of 4.5×10^{-40} C^2. m /N when you apply the field along the axis of the molecule, but only 2×10^{-40} for field perpendicular to this direction. When the field is at some angle to the axis, you must resolve it into parallel and perpendicular components, and multiply each by the pertinent polarizability:

$$\boldsymbol{p} = \alpha_{\perp}\boldsymbol{E}_{\perp} + \alpha_{\parallel}\boldsymbol{E}_{\parallel} \quad (2)$$

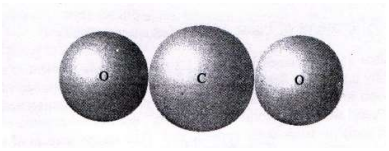

Figure: 21

In this case the induced dipole moment may not even be in the same direction as $\boldsymbol{E}$. And CO_2 is relatively simple, as molecules go, since the atoms at least arrange themselves in a straight line; for a completely asymmetrical molecule the equation (1) is replaced by the most general linear relation between $\boldsymbol{E}$ and $\boldsymbol{p}$:

$$\left.\begin{aligned} p_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ p_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ p_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z \end{aligned}\right\} \quad (3)$$

The set of nine constants α_{ij} Constitute the **polarizability tensor** for the molecule. Their actual values depend on the orientation of the axes you chose, though it is always possible to choose "principal" axes such that all the off-diagonal terms (α_{xy}, α_{zx} etc.) vanish, leaving just three nonzero polarizabilities: α_{xx}, α_{yy} and α_{xx}.

Example: 8 A primitive model for an atom consists on a point nucleus $(+q)$ surrounded by a uniformly charged spherical cloud $(-q)$ of radius a as shown in Figure: 22. Calculate the atomic polarizability of such atom.

Solution: In the presence of an external field $\boldsymbol{E}$, the nucleus will be shifted slightly to the right and electron cloud to the left see Figure: 23. (Because the actual displacements involved are extremely small, so it is reasonable to assume that the electron cloud retains its spherical shape.) Consider that the equilibrium occurs when the nucleus is displaced a distance d from the center of the sphere. At that point the external field pushing the nucleus to the right exactly balances the internal field pulling in to the left: $\boldsymbol{E} = \boldsymbol{E}_e$, where $\boldsymbol{E}_e$ is the field produced by the electron cloud. Now the field at a distance d from the center of a uniformly charged sphere is

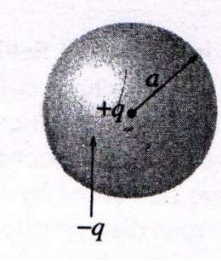

Figure: 22

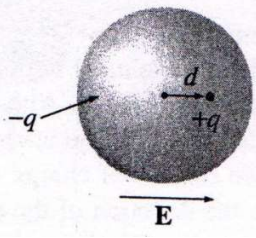

Figure: 23

$$E_e = \frac{1}{4\pi\varepsilon_0}\frac{qd}{a^3}$$

At equilibrium, then,

$$E = \frac{1}{4\pi\varepsilon_0}\frac{qd}{a^3},\ or \quad p = qd = (4\pi\varepsilon_0 a^3)E$$

The atomic polarizability is therefore,

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v$$

where v is the volume of the atom. Although this atomic model is extremely crude, the result equation (52) is not too bad-it's accurate to within a factor of four or so for many simple atoms.

3.3 Alignment of Polar molecules:

The neutral atom we discussed that had no dipole moment to start with – $\boldsymbol{p}$ was induced by the applied field. Some molecules have built-in, permanent dipole moments. In water molecule, for example, the electrons tend to cluster around the oxygen atom as shown in Figure: 24, and since the molecule is bent at 105^0, this leaves a negative charge at the vertex and a net positive charge at the opposite end. (The dipole moment of water is unusually large: 6.1×10^{-30} C .m; in fact, this is, what accounts for its effectiveness as a solvent.) What happens when such molecules (called polar molecules) are placed in an electric field?

If the field is uniform, the force on the positive end, $\boldsymbol{F}_+ = q\boldsymbol{E}$, exactly cancels the force on the negative end, $\boldsymbol{F}_- = -q\boldsymbol{E}$ as shown in Figure: 25.

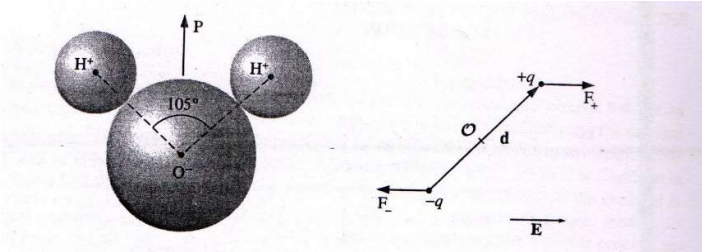

Figure: 24

Figure: 25

However, there will be a torque:

$$\boldsymbol{N} = \quad (\boldsymbol{r}_+ \times \boldsymbol{F}_+) + (\boldsymbol{r}_- \times \boldsymbol{F}_-) \quad (1)$$

$$= \left[\left(\frac{\boldsymbol{d}}{2}\right) \times (q\boldsymbol{E})\right] + \left[\left(-\frac{\boldsymbol{d}}{2}\right) \times (-q\boldsymbol{E})\right]$$

$$\boldsymbol{N} = \frac{q\boldsymbol{d}}{2} \times \boldsymbol{E} + \frac{q\boldsymbol{d}}{2} \times \boldsymbol{E} = q\boldsymbol{d} \times \boldsymbol{E} \quad (2)$$

Thus, a dipole $\boldsymbol{p} = q\boldsymbol{d}$ in a uniform field $\boldsymbol{E}$ experiences a torque

$$\boldsymbol{N} = \boldsymbol{p} \times \boldsymbol{E} \quad (3)$$

Note that $\boldsymbol{N}$ is in such a direction as to line $\boldsymbol{p}$ up parallel to $\boldsymbol{E}$, a polar molecule that is free to rotate will swing around until it points in the direction of the applied field.

If the field is nonuniform, so that $\boldsymbol{F}_+$ does not exactly balance $\boldsymbol{F}_-$, there will be a net force on the dipole, in addition to the torque. Of course, $\boldsymbol{E}$ must change rather abruptly for there to be significant variation in the space of one molecule, so this is not ordinarily a major consideration in discussing the behavior of dielectrics. Nevertheless, the formula for the force on a dipole in a nonuniform field is of some interest.

$$\boldsymbol{F} = \boldsymbol{F}_+ + \boldsymbol{F}_- = q(\boldsymbol{E}_+ + \boldsymbol{E}_-)$$

$$\boldsymbol{F} = q(\Delta \boldsymbol{E}) \quad (4)$$

Where $\Delta \boldsymbol{E}$ represents the difference between the field at the plus end and the field at the minus end. Assuming the dipole is very short,

we know that

$$dT = \left(\frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath} + \frac{\partial T}{\partial z}\hat{k}\right) \cdot \left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right)$$

$$dT = (\nabla T) \cdot (dl) \quad (5)$$

To approximate the small change in E_x:

$$\Delta E_x \equiv (\nabla E_x) \cdot \boldsymbol{d} \quad (6)$$

With corresponding formulas for E_y and E_z. More compactly,

$$\Delta \boldsymbol{E} = (\boldsymbol{d} \cdot \nabla)\boldsymbol{E} \quad (7)$$

And therefore,

$$\boldsymbol{F} = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E} \quad (8)$$

For a "perfect" dipole of infinitesimal length, the equation 54 gives the torque about the center of the dipole even in a nonuniform field; about any other point $\boldsymbol{N} = (\boldsymbol{p} \times \boldsymbol{E}) + (\boldsymbol{r} \times \boldsymbol{F})$.

3.4 Polarization:

When a piece of dielectric material is placed in an electric field, if the substance consists of neutral atoms (or nonpolar molecules), the field will induce in each a tiny dipole moment, pointing in the same direction as the field. If the material is made up of polar molecules, each permanent dipole will experience a torque, tending to line it up along the field direction. (Random thermal motion compete with this process, so the alignment is never complete, especially at higher temperatures, and disappears almost at once when the field is removed,)

Notice that these two mechanisms produce the same basic result: a lot of little dipoles pointing along the direction of the field –the material become **polarized**, A convenient measure of this effect is:

$$\boldsymbol{P} = \textit{ dipole moment per unit volume}$$

Which is called the **polarization**. In polar molecules there will be some polarization by displacement (though generally it is a lot easier to rotate a molecule than to stretch it, so second mechanism dominates.) it's even possible in some materials to "freeze in" polarization, so that it persists after the field is removed. But let's forget or a moment about the cause of the polarization and study the field that a chunk of polarized material itself produces: the original field, which was responsible for $\boldsymbol{P}$, plus the new field, which is due to $\boldsymbol{P}$.

4 The Field of a polarized object:

4.1 Bound Charges:

Consider a piece of polarized material, an object containing a lot of microscopic dipoles lined up. The dipole moment per unit volume $\boldsymbol{P}$ is given.

To find out field produced by this object (not the field that may have caused the polarization, but the field the polarization itself causes), we assume that the material is made up of infinitesimal dipoles and integrate to get the total field. It is easier to work with the potential. We know that for a single charge q, the potential is given by

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \quad (1)$$

for a single dipole $\boldsymbol{p}$, the potential is given by

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{p}}{r^2} \quad (2)$$

Where $\boldsymbol{r}$ is the vector from the dipole to the point at which we are evaluating the potential as shown in the Figure: 26. Here $\boldsymbol{p} = q\boldsymbol{r}$ or $\hat{\boldsymbol{r}}\cdot\boldsymbol{p} = qr$ or $q = \frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{p}}{r}$

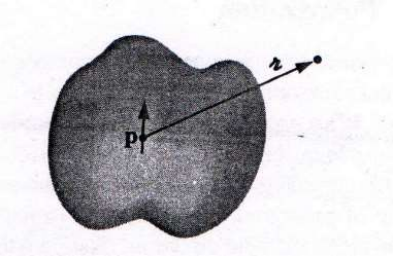

Figure: 26

As polarization is the dipole moment per unit volume: $\boldsymbol{P} = \boldsymbol{p}/d\tau'$, a dipole moment is $\boldsymbol{p} = \boldsymbol{P}d\tau'$ in each volume element dτ', so the total potential is

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{P}(\boldsymbol{r}')}{r^2}d\tau' \quad (3)$$

We know that:

$$\nabla'\left(\frac{1}{r}\right) = \frac{\hat{\boldsymbol{r}}}{r^2} \quad (4)$$

The differentiation is with respect to the source coordinates ($\boldsymbol{r}'$), Now equation (3) can be written as:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{P}(\boldsymbol{r}')}{r^2}d\tau' \quad (5)$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \boldsymbol{P}(\boldsymbol{r}')\cdot\left(\frac{\hat{\boldsymbol{r}}}{r^2}\right)d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \boldsymbol{P}\cdot\boldsymbol{\nabla}'\left(\frac{1}{r}\right)d\tau'$$

$$\therefore\ V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \boldsymbol{P}\cdot\boldsymbol{\nabla}'\left(\frac{1}{r}\right)d\tau' \quad (6)$$

We have product rule for divergence as

$$\nabla\cdot(f\boldsymbol{A}) = f(\nabla\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot(\nabla f) \quad (7)$$

By rearranging equation (7), we have

$$\boldsymbol{A}\cdot(\nabla f) = \nabla\cdot(f\boldsymbol{A}) - f(\nabla\cdot\boldsymbol{A}) \quad (8)$$

For present case, we have $\boldsymbol{A} = \boldsymbol{P}, f = 1/r$, therefore, equation (8) becomes

$$\boldsymbol{P}\cdot\nabla\left(\frac{1}{r}\right) = \nabla\cdot\left(\frac{\boldsymbol{P}}{r}\right) - \frac{1}{r}(\nabla\cdot\boldsymbol{P})$$

Therefore, equation (6) now becomes

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_V \boldsymbol{\nabla}'\cdot\left(\frac{\boldsymbol{P}}{r}\right)d\tau' - \int_V \frac{1}{r}(\boldsymbol{\nabla}'\cdot\boldsymbol{P})d\tau'\right] \quad (9)$$

Now using divergence theorem, we convert the first integral of equation (9): from volume integral to surface integral:

$$\int(\nabla\cdot\boldsymbol{v})d\tau = \oint \boldsymbol{v}\cdot d\boldsymbol{a} \quad (10)$$

$$\int_V \boldsymbol{\nabla}'\cdot\left(\frac{\boldsymbol{P}}{r}\right)d\tau' = \oint_S \frac{1}{r}\boldsymbol{P}\cdot d\boldsymbol{a}' \quad (11)$$

Therefore, equation (9) becomes

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{1}{r}\boldsymbol{P}\cdot d\boldsymbol{a}' - \frac{1}{4\pi\epsilon_0}\int_V \frac{1}{r}(\boldsymbol{\nabla}'\cdot\boldsymbol{P})d\tau' \quad (12)$$

The first term looks like the potential of a surface charge and

$$\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}} \quad (13)$$

Where $\hat{\boldsymbol{n}}$ is the normal unit vector and σ_b is the surface charge density. while the second term looks like the potential of a volume charge and

$$\rho_b = -\boldsymbol{\nabla}\cdot\boldsymbol{P} \quad (14)$$

where ρ_b is the volume charge density.

With these definitions, equation (12) becomes

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\sigma_b}{r}\,da' \quad + \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_b}{r}\,d\tau' \quad (15)$$

Equation (15) shows that the potential (and hence the field also) of a polarized object is the same as that produced by a volume charge density $\rho_b = -\boldsymbol{\nabla}\cdot\boldsymbol{P}$ plus surface charge density $\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}}$ Instead of integrating the contributions of all the infinitesimal dipoles, as in equation (3), we just find those **bound charges**, and then calculate the fields they produce, in the same way we calculate the field of any other volume and surface charges (for example, using Gauss's law.)

Example: 9 Find the electric field produced by a uniformly polarized sphere of radius R.

Solution: We may choose the z axis to coincide with the direction of polarization as shown in the Figure: given below. The volume bound charge density ρ_b is zero, since $\boldsymbol{P}$ is uniform, but

$$\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}} = P\cos\theta \quad (1)$$

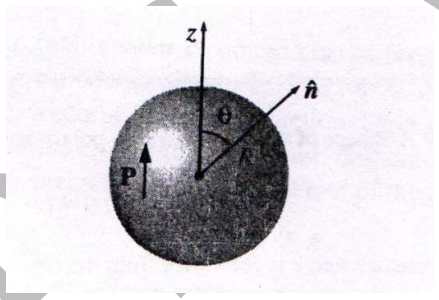

Figure: 27

Where θ is the usual spherical coordinate. We want to find out the field produced by a charge density $P\cos\theta$ plastered over the surface of a sphere. But we have already computed the potential of such a configuration

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta\,, & for\ r \le R, \\ \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta\,, & for\ r\ \ge R. \end{cases} \quad (2)$$

Since $r\cos\theta = z$ the field inside the sphere is uniform.

$$E = -\nabla V = -\frac{P}{3\epsilon_0}\hat{\boldsymbol{z}} = -\frac{1}{3\epsilon_0}\boldsymbol{P},\ for\ r\ < R \quad (3)$$

This remarkable result will be very useful. Outside the sphere the potential is identical to that of a perfect dipole at the origin.

$$V = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2} \quad for\ r \geq R \qquad (4)$$

Whose dipole moment is, not surprisingly, equal to the total dipole moment of the sphere.

$$\boldsymbol{p} = \frac{4}{3}\pi R^3 \boldsymbol{P} \qquad (5)$$

The field of the uniformly polarized sphere is as shown in the Figure: (28) given below.

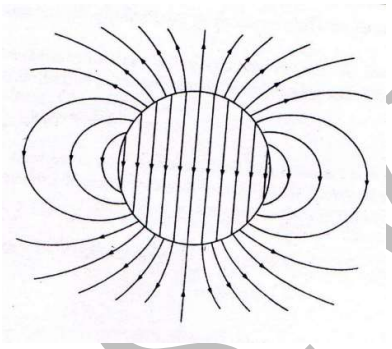

Figure: 28

4.2 Physical interpretation of Bound Charges:

We found that the field of a polarized object is identical to the field that would be produced by a certain distribution of "bound charges" σ_b *and* ρ_b. But this conclusion emerged in the course of abstract manipulations on the integral as in this equation

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{p}(\boldsymbol{r}')}{r^2}d\tau' \qquad (1)$$

and left us with no clue as to the physical meaning of these bound charges. Some authors show that the bound charges are in some sense 'fictitious' –used to facilitate the calculation of fields. σ_b *and* ρ_b represents perfectly genuine accumulations of charge. In this we will see that how polarization leads to such accumulations of charges.

Suppose we have a long string of dipoles, as shown in the Figure: 29. Along the line, the head of one effectively cancels the tail of its neighbor, but at the ends there are two charges left over: plus at the right end and minus at the left. It is as if we had peeled off an electron at one end and carried it all the way down to the other end, though in fact no single electron made the whole

trip –a lot of tiny displacements add up to one large one. The net charge at the ends bound charge, it cannot be removed; in a dielectric every electron is attached to a specific atom or molecule. But apart from that, bound charge is no different from any other kind.

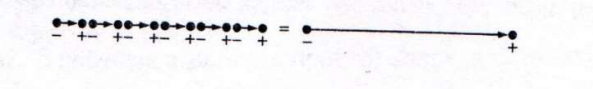

Figure: 29

To calculate the actual amount of bound charge resulting from a given polarization, examine a "tube" of dielectric parallel to $\boldsymbol{P}$. The dipole of the tiny chunk shown in Figure: 30 is $P(Ad)$, where A is the cross-sectional area of the tube and d is the length of the chunk. In terms of the charge (q) at the end, the same dipole moment can be written as qd. The bound charge that piles up at the right end of the tube is therefore,

$$q = PA \quad (1)$$

If the ends have been sliced off perpendicularly, the surface charge density is

$$\sigma_b = \frac{q}{A} = P \quad (2)$$

For an oblique cut as shown in Figure: 31, the charge is still the same, but $A = A_{end}\, cos\, \theta$. So

$$\sigma_b = \frac{q}{A_{end}} = P\, cos\, \theta = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} \quad (3)$$

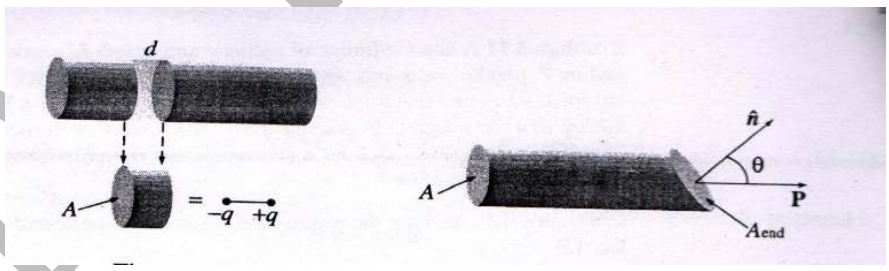

Figure: 30 Figure: 31

The effect of the polarization then, is to paint a bound charge $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}$ over the surface of the material.

If the polarization is nonuniform we get accumulations of bound charge within the material as well as on the surface. Consider Figure: 32 it suggests that a diverging P results in a pileup of negative charge. Indeed, the net bound charge $\int \rho_b d\tau$ in a given volume is equal and opposite

to the amount that has been pushed out through the surface. The latter (by the same reasoning we used before) is $\boldsymbol{P} \cdot \hat{\boldsymbol{n}}$ per unit area.

$$\int_v \rho_b \, d\tau = -\oint_S \boldsymbol{P} \cdot d\boldsymbol{a} = -\int_v (\boldsymbol{\nabla} \cdot \boldsymbol{P}) \, d\tau \quad (4)$$

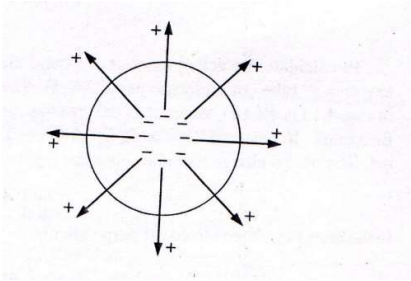

Figure: 32

Since this is true for any volume, we have

$$\rho_b = -\boldsymbol{\nabla} \cdot \boldsymbol{P} \quad (5)$$

Confirming, again, the more rigorous conclusion in the topic bound charge.

Example: 10 There is another way of analyzing the uniformly polarized sphere, (the previous example), which nicely illustrates the idea of a bound charge. Consider two spheres of charge; a positive sphere and a negative sphere. Without polarization the two are superimposed and cancel completely. But when the material is uniformly polarized, all the plus charges move slightly up ward (the z direction) and all minus charges move slightly downward. See Figure: 33. Two spheres no longer overlap perfectly: at the top there is a "cap" of leftover positive charge and at the bottom a "cap" of negative charge.

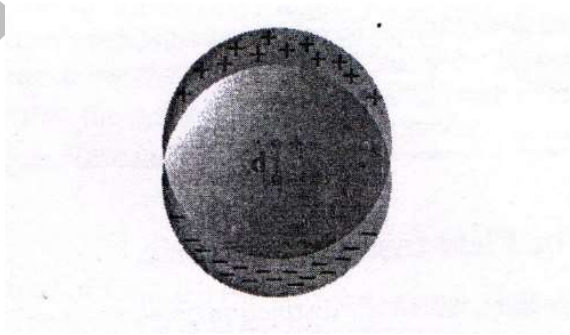

Figure: 33

This "leftover" charge is precisely the bound surface charge σ_b.

The field in the region of overlap between two uniformly charged sphere is given as

$$\boldsymbol{E} = -\frac{1}{4\pi\epsilon_0}\frac{q\boldsymbol{d}}{R^3} \quad (1)$$

Where q is the total charge of the positive sphere, d is the vector from the negative center to the positive center, and R is the radius of the sphere. We can express this in terms of the polarization of the sphere, as polarization is the dipole moment per unit volume,

$$\boldsymbol{p} = q\boldsymbol{d} = \left(\frac{4}{3}\pi R^3\right)\boldsymbol{P} \quad (2)$$

as

$$\boldsymbol{E} = -\frac{1}{4\pi\epsilon_0}\frac{q\boldsymbol{d}}{R^3} = -\frac{1}{4\pi\epsilon_0}\left(\frac{4}{3}\pi R^3\right)\boldsymbol{P} = -\frac{1}{3\epsilon_0}\boldsymbol{P}$$

$$\boldsymbol{E} = -\frac{1}{3\epsilon_0}\boldsymbol{P} \quad (3)$$

Meanwhile the points outside, it is as though all the charges on each sphere were concentrated at the respective center. We have, then, a dipole, with potential

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2} \quad (4)$$

Remember that $\boldsymbol{d}$ is some small fraction of an atomic radius.

4.3 The field inside a Dielectric:

Consider a pure dipole, the potential of a pure dipole is given as

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2} \quad (1)$$

We presumed to represent discrete molecular dipoles by a continuous density function $\boldsymbol{P}$. Outside the dielectric, we are far away from the molecules (r is many times greater than the separation distances between plus and minus charges), so the dipole potential dominates overwhelmingly and the detailed "graininess" of the source is blurred by distance. Inside the dielectric, however, we can hardly pretend to be far from all the dipoles.

The electric field inside matter must be complicated on the microscopic level. If you happen to be near an electron, the field is gigantic, whereas a short distance away it may be small or point

in a totally different direction. Moreover, an instant later, as the atoms move about, the field will have altered entirely. This true **microscopic** field would be utterly impossible to calculate. Just as, for macroscopic purposes, we regard water as continuous fluid, ignoring its molecular structure, so also we can ignore the microscopic bumps and wrinkles in the electric field inside matter, and concentrate on the **macroscopic** field.

This is defined as the average field over regions large enough to contain many thousands of atoms (so that the uninteresting microscopic fluctuations are smoothed over), and yet small enough to ensure that we do not wash out any significant large-scale variations in the field. (In practice, this means we must average over regions much smaller than the dimensions of the object itself.) ordinarily, the macroscopic field is the field inside matter.

Suppose we want to calculate the macroscopic field at some point r within a dielectric as shown in Figure: 34.

We know we must average the true (microscopic) field over an appropriate volume, so let us draw a small sphere about $\boldsymbol{r}$, of radius, say, a thousand times the size of a molecule. The macroscopic field at $\boldsymbol{r}$, then, consists of two parts: the average field over the sphere due to all charges outside, plus the average due to all charges inside.

$$\boldsymbol{E} = \boldsymbol{E}_{out} + \boldsymbol{E}_{in} \quad (1)$$

Figure: 34

Now we also know that the average field (over a sphere), produced by charge outside, is equal to the field they produce at the center, so $\boldsymbol{E}_{out}$ is the field at $\boldsymbol{r}$ due to the dipoles exterior to the sphere. These are far enough away that we can safely use the equation,

$$V_{out} = \frac{1}{4\pi\epsilon_0}\int_{outside} \frac{\hat{\boldsymbol{r}} \cdot \boldsymbol{P}(\boldsymbol{r}')}{r^2} d\tau' \quad (2)$$

The dipole inside the sphere are too close to treat in this fashion. But fortunately, all we need is their average field, and can be given as

$$\boldsymbol{E}_{in} = -\frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}}{R^3} \quad (3)$$

As the average field inside a sphere of radius R, due to all the charges within the sphere is

$$\boldsymbol{E}_{ave} = -\frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}}{R^3} \quad (4)$$

regardless of the details of the charge distribution within the sphere. The only relevant quantity is the total dipole moment,

$$\boldsymbol{p} = \left(\frac{4}{3}\pi R^3\right)\boldsymbol{P} \quad (5)$$

$$\boldsymbol{E}_{in} = -\frac{1}{3\epsilon_0}\boldsymbol{P} \quad (6)$$

Now, by assumption the sphere is small enough that P does not vary significantly over its volume, so the term left out of integral in equation (02) corresponds to the field at the center of a uniformly polarized sphere: $-\frac{1}{3\epsilon_0}\boldsymbol{P}$. But this is precisely what $\boldsymbol{E}_{in}$ in equation (6) puts back in. The macroscopic field, then, is given by the potential

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{P}(\boldsymbol{r}')}{r^2}d\tau' \quad (7)$$

Where the integral runs over the entire volume of the dielectric. This is, of course, what we used (bound charges) earlier; without realizing it, we were correctly calculating the averaged, macroscopic field, for points inside the dielectric.

Notice that it all revolves around the curious fact that the average field over any sphere (due to charge inside) is the same as the field at the center of a uniformly polarized sphere with the same total dipole moment. This means that no matter how crazy the actual microscopic charge configuration, we can replace it by a nice smooth distribution of perfect dipoles, if all we want is the macroscopic (average) field.

The macroscopic field is certainly independent of the geometry of the average region. Presumably, one could reproduce the same argument for a cube or an ellipsoid or whatever---the calculation might be more difficult, but the conclusion would be the same.

5 The Electric displacement:

5.1 Gauss's law in the presence of Dielectrics:

We understand that the effect of polarization is to produce the accumulations of bound charges, $\rho_b = -\boldsymbol{\nabla} \cdot \boldsymbol{P}$ within the dielectric and $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}$ on the surface. The field due to polarization of the medium is just the field of this bound charge.

Now consider the field attributable to the bound charge plus the field due to everything else (i.e., **free charge**). The free charge might consist of electrons on a conductor or ions embedded in the dielectric material or whatever; any charge, in the other words, "free charge" is not a result of polarization.

Within the dielectric, the total charge density can be given as:

$$\rho = \rho_b + \rho_f \quad (1)$$

And Gauss's law $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\epsilon_0$

$$\therefore \ \epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho = \rho_b + \rho_f = -\boldsymbol{\nabla} \cdot \boldsymbol{P} + \rho_f \quad (2)$$

Where $\boldsymbol{E}$ is now the total field, due to polarization and due to free charge.

By combine the two divergence terms of equation (2):

$$\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} + \boldsymbol{\nabla} \cdot \boldsymbol{P} = \rho_f$$

$$\boldsymbol{\nabla} \cdot (\epsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \rho_f \quad (3)$$

The expression is parentheses is known as **electric displacement** ($\boldsymbol{D}$).

$$\boldsymbol{D} = \boldsymbol{\epsilon_0 E} + \boldsymbol{P} \quad (4)$$

In terms of $\boldsymbol{D}$, the Gauss's law: equation (3) becomes

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f \quad (5)$$

To derive an integral form, take volume integral of equation (5)

$$\int \boldsymbol{\nabla} \cdot \boldsymbol{D} \, d\tau = \int \rho_f \, d\tau \quad (6)$$

As $\rho_f = \frac{Q_{fenc}}{d\tau}$, $\int \boldsymbol{\nabla} \cdot \boldsymbol{D}\, d\tau = \oint \boldsymbol{D} \cdot d\boldsymbol{a}$, equation (6) becomes

$$\oint \boldsymbol{D} \cdot d\boldsymbol{a} = Q_{fenc} \quad (7)$$

Where Q_{fenc} denotes the total free charge enclosed in the volume. This is a particularly useful way to express Gauss's law, in the contests of dielectrics, because it makes reference only to free charges, and free charge is the stuff we control. Bound charge comes along for the ride: when we put free charge in place, a certain polarization be there and this polarization produces the bound charge. Initially we know ρ_f, but we do not know ρ_b. In particular, whenever the requisite symmetry is present, we can immediately calculate $\boldsymbol{D}$ by the standard Gauss's law methods.

Here we left out the surface bound charge σ_b in deriving equation $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f$, because we cannot apply Gauss's law precisely at the surface of a dielectric, for here ρ_b blows up, taking the divergence of E with it. But if we consider that the edge of the dielectric having some finite thickness within which the polarization tapers off to zero then there is no surface bound charge; ρ_b varies rapidly but smoothly within this "skin". And Gauss's can be safely applied everywhere.

Example: 11 A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a, as shown in the Figure: 35. Find the electric displacement $\boldsymbol{D}$.

Solution: Draw a cylindrical Gaussian surface, of radius s and length L as shown in the Figure: and apply the equation

$$\oint \boldsymbol{D} \cdot d\boldsymbol{a} = Q_{fenc} \quad (1)$$

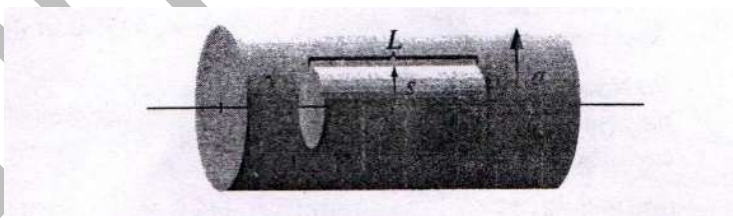

Figure: 35

Surface area of this cylinder is $2\pi sL$, line charge density is λ, therefore $Q_{fenc} = \lambda L$.

$$D(2\pi sL) = \lambda L \quad (2)$$

Therefore,

$$\boldsymbol{D} = \frac{\lambda}{2\pi s}\hat{\boldsymbol{s}} \quad (3)$$

This formula holds both within the insulation and outside it. In the latter region, $\boldsymbol{P} = 0$, so

$$\boldsymbol{E} = \frac{1}{\epsilon_0}\boldsymbol{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\boldsymbol{s}}, \quad for\ s > a \quad (4)$$

Inside the rubber the electric field cannot be determined, since we do not know $\boldsymbol{P}$.

5.2 Deceptive Parallel:

(i) The equation $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_f$, looks just like Gauss's law: $\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho/\epsilon_0$, only the total charge density ρ is replaced by the free charge density ρ_f , and $\boldsymbol{D}$ is substituted for $\epsilon_0\boldsymbol{E}$. For this reason, if we say that: $\boldsymbol{D}$ is "just like" $\boldsymbol{E}$ (apart from the factor ϵ_0), except that its source of ρ_f instead of ρ or "To solve problems involving dielectrics, we just forget all about the bound charges-calculate the field, only call the answer $\boldsymbol{D}$ instead of $\boldsymbol{E}$". This reasoning is seductive, but the conclusion is false.

(ii) In particular, there is no "Coulomb's law" for $\boldsymbol{D}$.

$$\boldsymbol{D}(\boldsymbol{r}) \neq \frac{1}{4\pi}\int \frac{\hat{\boldsymbol{r}}}{r^2}\rho_f(\boldsymbol{r}')d\tau'$$

The parallel between $\boldsymbol{E}$ and $\boldsymbol{D}$ is more subtle than that.

(iii) To determine a vector field, the divergence alone is insufficient; we need to know the curl as well. One tends to forget this in the case of electrostatic fields because curl of $\boldsymbol{E}$**:** $(\boldsymbol{\nabla}\times\boldsymbol{E})$ is always zero. But the curl of $\boldsymbol{D}$**:** $(\boldsymbol{\nabla}\times\boldsymbol{D})$ is not always zero.

$$\boldsymbol{\nabla}\times\boldsymbol{D} = \epsilon_0(\boldsymbol{\nabla}\times\boldsymbol{E}) + (\boldsymbol{\nabla}\times\boldsymbol{P}) = \boldsymbol{\nabla}\times\boldsymbol{P}$$

And there is no reason, in general, to suppose that the curl of $\boldsymbol{P}$ vanishes. Sometime it does, but more often it does not.

(iv) Suppose, there is no free charge anywhere, so we believe that the only source of $\boldsymbol{D}$ is ρ_f, and as $\rho_f = 0 \therefore \boldsymbol{D} = 0$ everywhere, and hence that $\boldsymbol{E} = (-1/\epsilon_0)\boldsymbol{P}$ inside and $\boldsymbol{E} = 0$ outside the electret, which is obviously wrong. Because $\boldsymbol{\nabla}\times\boldsymbol{D} \neq 0$.

(v) moreover, $\boldsymbol{D}$ cannot be expressed as the gradient of scalar –there is no "potential" for $\boldsymbol{D}$.

(vi) When we have to compute the electric displacement, first look for symmetry. If problem exhibits spherical, cylindrical or plane symmetry, then we can get $\boldsymbol{D}$ directly using the equation $\oint \boldsymbol{D}\cdot d\boldsymbol{a} = Q_{fenc}$ by the usual Gauss's law methods. (Evidently in such cases

$\boldsymbol{\nabla} \times \boldsymbol{P}$ is automatically zero.) If the requisite symmetry is absent, we have to think of another approach and, in particular, we must not assume that $\boldsymbol{D}$ is determined exclusively by the free charges.

5.3 Boundary Conditions:

The electrostatic boundary conditions can be written in terms of $\boldsymbol{D}$.

$\oint \boldsymbol{D} \cdot d\boldsymbol{a} = Q_{fenc}$ tells the discontinuity in the component perpendicular to an interface;

$$\boldsymbol{D}^{\perp}_{above} - \boldsymbol{D}^{\perp}_{below} = \sigma_f \quad (1)$$

While equation:

$$\boldsymbol{\nabla} \times \boldsymbol{D} = \epsilon_0(\boldsymbol{\nabla} \times \boldsymbol{E}) + (\boldsymbol{\nabla} \times \boldsymbol{P}) = \boldsymbol{\nabla} \times \boldsymbol{P} \quad (\mathbf{2})$$

gives the discontinuity in parallel components:

$$\boldsymbol{D}^{\parallel}_{above} - \boldsymbol{D}^{\parallel}_{below} = \boldsymbol{P}^{\parallel}_{above} - \boldsymbol{P}^{\parallel}_{below} \quad (3)$$

In the presence of dielectrics these are sometimes more useful than the corresponding boundary conditions on $\boldsymbol{E}$:

$$\boldsymbol{E}^{\perp}_{above} - \boldsymbol{E}^{\perp}_{belo} = \frac{1}{\epsilon_0}\sigma \quad (4)$$

and

$$\boldsymbol{E}^{\parallel}_{above} - \boldsymbol{E}^{\parallel}_{below} = 0 \quad (5)$$